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Equivariantly slicing strongly negative amphichiral knots

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We prove obstructions to a strongly negative amphichiral knot bounding an equivariant slice disk in the 4–ball using the determinant, Spin^c–structures and Donaldson's theorem. Of the 16 slice strongly negative amphichiral knots with 12 or fewer crossings, our obstructions show that 8 are not equivariantly slice, we exhibit equivariant ribbon diagrams for 5 others, and the remaining 3 are unknown. Finally, we give an obstruction to a knot being strongly negative amphichiral in terms of Heegaard Floer correction terms.

57K10, 57M60

1 Introduction

A *strongly negative amphichiral* knot (K,σ) is a smooth knot $K \subset S^3$ along with a smooth (orientation-reversing) involution $\sigma\colon S^3 \to S^3$ such that $\sigma(K) = K$ and σ has exactly two fixed points, both of which lie on K; see Figure 1. A knot $K \subset S^3$ is *slice* if it bounds a smooth disk (the *slice disk*) properly embedded in B^4. Our main goal is to study when there exists an equivariant slice disk for a strongly negative amphichiral knot (K,σ). Specifically, we are interested in the following property:

Definition 1.1 A strongly negative amphichiral knot (K,σ) is *equivariantly slice* if there is a smooth slice disk D and a smooth involution $\sigma'\colon B^4 \to B^4$ with $\sigma'(D) = D$ which restricts to σ on $\partial B^4 = S^3$.

Figure 1 gives an example of a strongly negative amphichiral diagram, that is, a knot diagram with the strongly negative amphichiral symmetry given by π–rotation around an axis perpendicular to the page followed by reflection across the plane of the diagram. Furthermore, the knot in Figure 1 is equivariantly slice. The slice disk is given by performing the pair of equivariant band moves shown in red, then equivariantly capping off the resulting 3–component unlink in B^4. Among nontrivial prime knots with 12 or fewer crossings, there are 16 slice strongly negative amphichiral knots. For five of them, namely 8_9, 10_{99}, $12a_{819}$, $12a_{1269}$ and $12n_{462}$, we found similar equivariant ribbon diagrams; see the table in Section 7.

Strongly negative amphichiral knots, and in particular the equivariant surfaces they bound in the 4–ball, have been studied less than their more popular orientation-preserving cousins: strongly invertible knots (see for example Boyle and Issa [2] and Sakuma [23]) and periodic knots (see for example [2], Cha and Ko [5], Davis and Naik [6], and Grove and Jabuka [14] among others). Many of the obstructions used

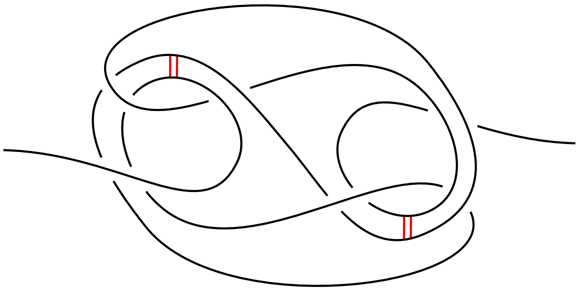

Figure 1: A strongly negative amphichiral diagram for 8_9. The symmetry is given by π–rotation around an axis perpendicular to the page, followed by a reflection across the plane of the diagram. An equivariant slice disk can be seen by performing the band moves shown in red.

in the strongly invertible and periodic settings do not adapt to the strongly negative amphichiral case. In fact, even showing that the (nonequivariant) 4–genus for strongly negative amphichiral knots can be arbitrarily large was only recently accomplished by Miller [20].

Our first equivariant slice obstruction comes from studying the knot determinant. It was shown by Goeritz [10] that the determinant of an amphichiral knot is the sum of two squares (see also Friedl, Miller and Powell [9] for a partial generalization and Stoimenow [24] for the converse). We prove the following strengthening of this determinant condition in the case that K bounds an equivariant slice disk:

Theorem 1.2 *If K is an equivariantly slice strongly negative amphichiral knot*, *then* $\det(K)$ *is the square of a sum of two squares.*

Theorem 1.2 shows that the six slice strongly negative amphichiral knots 10_{123}, $12a_{435}$, $12a_{990}$, $12a_{1019}$, $12a_{1225}$ and $12n_{706}$ are not equivariantly slice.

Our second obstruction, which applies to knots with an alternating strongly negative amphichiral diagram, comes from applying Donaldson's theorem [8]. Donaldson's theorem can often be used to obstruct the existence of slice disks (see for example Lisca [18]). More recently, it has also been used to obstruct equivariant slice disks for strongly invertible and periodic knots [2]. A key ingredient in that obstruction is the existence of an invariant definite spanning surface for the knot. In contrast, strongly negative amphichiral knots do not bound invariant spanning surfaces in S^3. Instead, we use the fact that, if K bounds an equivariant slice disk D, then the subset S of Spin^c–structures on the double branched cover $Y = \Sigma(S^3, K)$ that extend over $\Sigma(B^4, D)$ is $\tilde{\sigma}$–invariant, where $\tilde{\sigma}$ is a lift of the symmetry σ to Y; see Proposition 4.1 and the discussion following its proof. Donaldson's theorem can be used to obtain restrictions on S. Using the interplay between the pair of checkerboard surfaces exchanged by the symmetry, we carefully keep track of Spin^c–structures, allowing us to compute the $\tilde{\sigma}$–action on $\mathrm{Spin}^c(Y)$. This results in a nice combinatorial description of the $\tilde{\sigma}$–action on $\mathrm{Spin}^c(Y)$ in terms of the oriented incidence matrices of the checkerboard graphs for an alternating symmetric diagram. Specifically, we prove the following theorem:

Theorem 1.3 *Let (K,σ) be a knot with an alternating strongly negative amphichiral diagram and let $Y=\Sigma(S^3,K)$. Let $F_\pm$ be the positive and negative definite checkerboard surfaces, let $J^*_\pm$ be compatible oriented incidence matrices with a row removed*[1] *for the checkerboard graphs of $F_\pm$, and let $A_\pm = J^*_\pm (J^*_\pm)^\mathsf{T} \in M_n(\mathbb{Z})$ be the Goeritz matrices for $F_\pm$. Then there is a lift $\tilde\sigma\colon Y\to Y$ for which the map $\tilde\sigma^*\colon \mathrm{Spin}^c(Y)\to \mathrm{Spin}^c(Y)$ is determined by*

$$\tilde\sigma^*[J^*_+ v] = [J^*_- v] \quad \textit{for all } v\in\mathbb{Z}^{2n} \textit{ with } v\equiv (1,1,\dots,1)^\mathsf{T}\in(\mathbb{Z}/2\mathbb{Z})^{2n},$$

where $\mathrm{Spin}^c(Y)\cong \mathrm{Char}(\mathbb{Z}^n, A_+)/\mathrm{im}(2A_+)$. Moreover, if K is equivariantly slice, then there is a lattice embedding $A\colon(\mathbb{Z}^n,A_+)\to(\mathbb{Z}^n,\mathrm{Id})$ such that

$$S=\{[u]\in\mathrm{Spin}^c(Y)\mid u=A^\mathsf{T} v \textit{ for some } v\in\mathbb{Z}^n \textit{ with } v\equiv(1,1,\dots,1)^\mathsf{T}\in(\mathbb{Z}/2\mathbb{Z})^n\}$$

is $\tilde\sigma^*$–invariant.

Using Theorem 1.3, we show that $12a_{1105}$ and $12a_{1202}$ are not equivariantly slice (see Section 5), even though they satisfy the determinant condition in Theorem 1.2 as $\det(12a_{1105}) = 17^2 = (4^2+1^2)^2$ and $\det(12a_{1202}) = 13^2 = (3^2+2^2)^2$. Of the slice strongly negative amphichiral knots with 12 or fewer crossings, this leaves only $12a_{458}$, $12a_{477}$ and $12a_{887}$ for which equivariant sliceness is unknown. See Section 7 for a table of equivariant knot diagrams for these knots.

Our analysis of the $\tilde\sigma$–action on $\mathrm{Spin}^c(\Sigma(S^3,K))$ also leads us to the following obstruction to strongly negative amphichirality in terms of Heegaard Floer correction terms.

Theorem 1.4 *Let (K,σ) be a strongly negative amphichiral knot and let $\tilde\sigma$ be a lift of σ to $Y:=\Sigma(S^3,K)$* (see Proposition 2.1). *Then the orbits of $\mathrm{Spin}^c(Y)$ under the action of $\tilde\sigma$ take the following form*:

(1) *There is exactly one orbit $\{\mathfrak{s}_0\}$ of order 1 with $d(Y,\mathfrak{s}_0)=0$.*

(2) *All other orbits $\{\mathfrak{s},\tilde\sigma(\mathfrak{s}),\tilde\sigma^2(\mathfrak{s}),\tilde\sigma^3(\mathfrak{s})\}$ have order 4 and*

$$d(Y,\tilde\sigma^i(\mathfrak{s})) = (-1)^i d(Y,\mathfrak{s}) \quad \textit{for all } i.$$

For example, the figure eight knot 4_1 is strongly negative amphichiral and $\Sigma(S^3,4_1)=L(5,2)$, which has correction terms $\left\{0,\frac{2}{5},-\frac{2}{5},\frac{2}{5},-\frac{2}{5}\right\}$. We checked that, for all 2–bridge knots with 12 or fewer crossings, the d–invariants have this structure precisely when the knot is strongly negative amphichiral, leading us to the following conjecture:

Conjecture 1.5 *Let $p,q\in\mathbb{N}$ with p odd and $(p,q)=1$. The following are equivalent*:

(1) *The Heegaard Floer correction terms of the lens space $L(p,q)$ can be partitioned into multisets, each of the form $\{r,-r,r,-r\}$ for some $r\in\mathbb{Q}$, and a single set $\{0\}$.*

[1] See Definition 4.6. Here $J^*_\pm$ is an n by $2n$ matrix.

(2) *The* 2*–bridge knot* $K(p/q)$ *is amphichiral.*

(3) *There is an orientation-reversing self-diffeomorphism of* $L(p,q)$*.*

(4) $q^2 \equiv -1 \pmod p$.

We note that (2), (3) and (4) are known to be equivalent (see for example Bonahon [1, Theorem 3], Hodgson and Rubinstein [15, Corollary 4.12] and Stoimenow [24, Section 4]). Theorem 1.4 shows that (2) implies (1), since $\Sigma(S^3, K(p/q)) = L(p,q)$ and a 2–bridge knot is amphichiral if and only if it is strongly negative amphichiral. Thus Conjecture 1.5 is equivalent to showing that (1) implies any of the other conditions.

1.1 Open questions

We conclude the introduction with a list of interesting open questions for further exploration.

Question 1.6 Is there a nonslice strongly negative amphichiral knot with equivariant 4–genus larger than its 4–genus?

Question 1.7 Is there a strongly negative amphichiral knot which is topologically equivariantly slice but not smoothly equivariantly slice?

Question 1.8 Is every strongly negative amphichiral knot with Alexander polynomial 1 topologically equivariantly slice?

Question 1.9 If a strongly negative amphichiral knot is smoothly equivariantly slice, then must the knot admit an equivariant ribbon diagram, as in Figure 1?

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2 Lifting the action to the double branched cover

In this section we show that the strongly negative amphichiral involution σ on S^3 lifts to the double branched cover $\Sigma(S^3, K)$. Since we are interested in equivariant slice disks for K, we also show that this lift $\tilde{\sigma}$ can be extended to $\Sigma(B^4, S)$ for any equivariant surface $S \subset B^4$ with $\partial S = K$. Specifically, we have the following proposition, which is similar to [2, Proposition 12]. However, in our situation there are no fixed points disjoint from the branch set; the amphichiral involution lifts to an order-4 symmetry on the double branched cover.

Proposition 2.1 *Let* $S \subset S^4$ *be a closed connected smoothly embedded surface and let* $\sigma\colon (S^4, S) \to (S^4, S)$ *be a smooth involution with nonempty fixed-point set contained in* S*. Let* $p\colon \Sigma(S^4, S) \to S^4$ *be the projection map from the double branched cover and let* $\tau\colon \Sigma(S^4, S) \to \Sigma(S^4, S)$ *be the nontrivial* *deck transformation map. Then there is a lift* $\tilde{\sigma}\colon \Sigma(S^4,S)\to\Sigma(S^4,S)$ *such that the following diagram commutes*:

$$\begin{tikzcd} \Sigma(S^4,S) \arrow[r, "\tilde{\sigma}"] \arrow[d, "p"'] & \Sigma(S^4,S) \arrow[d, "p"] \\ S^4 \arrow[r, "\sigma"] & S^4 \end{tikzcd}$$

Furthermore, $\tilde{\sigma}^2=\tau$, *and there are exactly two such lifts*, *namely* $\tilde{\sigma}$ *and* $\tilde{\sigma}^3$.

Proof Let $N(S)$ be an equivariant tubular neighborhood of S and $E=S^4\backslash N(S)$ be the surface exterior. Denote by $\widetilde{E}$ the double cover of E corresponding to the kernel G of $\pi_1(E)\to H_1(E;\mathbb{Z}/2\mathbb{Z})$. We also choose a basepoint $s\in E$ and lifts $\tilde{s},\tilde{t}\in\widetilde{E}$ with $p(\tilde{s})=s$ and $p(\tilde{t})=\sigma(s)$.

Since G is the unique index-2 subgroup of $\pi_1(E,s)$, it is a characteristic subgroup. Hence G is also the image of $\pi_1(\sigma\circ p)\colon \pi_1(\widetilde{E},\tilde{t})\to\pi_1(E,s)$. Then by the covering space lifting property, since $\operatorname{im}(\pi_1(\sigma\circ p))\subseteq\operatorname{im}(\pi_1(p)\colon\pi_1(\widetilde{E},\tilde{s})\to\pi_1(E,s))$, there is a unique map $\tilde{\sigma}\colon(\widetilde{E},\tilde{t})\to(\widetilde{E},\tilde{s})$ such that $p\circ\tilde{\sigma}=\sigma\circ p$. By the equivariant tubular neighborhood theorem [3, Theorem VI.2.2], ∂E can be identified with the unit normal bundle of S, where σ preserves S^1 fibers. Lifting this bundle structure to $\partial\widetilde{E}$, p gives a bijection between the set of fibers of ∂E and the set of fibers of $\partial\widetilde{E}$ (p restricts to a two-to-one covering on each fiber). In particular, $\tilde{\sigma}$ preserves the set of S^1 fibers on the S^1–bundle boundary of $\widetilde{E}$. By extending this action over each D^2 fiber, we can (smoothly) extend $\tilde{\sigma}$ to the tubular neighborhood $p^{-1}(N(S))\subset\Sigma(S^4,S)$ such that $p\circ\tilde{\sigma}=\sigma\circ p$.

Finally, $p\circ\tilde{\sigma}=\sigma\circ p$ implies that $p\circ\tilde{\sigma}^2=\sigma^2\circ p=p$, so that $\tilde{\sigma}^2$ is either the identity map, or else the nontrivial deck transformation τ on $\Sigma(B^4,S)$. Note that, in either case, $\tilde{\sigma}^4$ is the identity map. However, σ acts by π–rotation on an equivariant meridian α of a fixed point of σ. Indeed, if σ acted by reflection or identity on α, then there would be fixed points disjoint from S. In the branched cover we then have that $\tilde{\sigma}$ acts by $\frac{\pi}{2}$–rotation on $p^{-1}(\alpha)$. Thus $\tilde{\sigma}$ has order 4 and $\tilde{\sigma}^2=\tau$, as desired. Finally, we note that there are exactly two lifts, $\tilde{\sigma}$ and $\tau\circ\tilde{\sigma}=\tilde{\sigma}^3$, one for each choice of $\tilde{t}$. □

Corollary 2.2 *Let* (K,σ) *be a strongly negative amphichiral knot with double branched cover* $\Sigma(S^3,K)$. *Let* $S\subset B^4$ *be a smooth properly embedded surface with boundary* K *which is invariant under an extension of* σ *to* B^4 (*which we again call* σ). *Then there is a lift* $\tilde{\sigma}\colon\Sigma(B^4,S)\to\Sigma(B^4,S)$ *such that* $\tilde{\sigma}^2=\tau$ (*and hence* $\tilde{\sigma}^4=\mathrm{Id}$) *and* $p\circ\tilde{\sigma}=\sigma\circ p$. *In fact*, *there are exactly two such lifts*, *namely* $\tilde{\sigma}$ *and* $\tilde{\sigma}^3$.

Proof Take the double of $\Sigma(B^4,S)$ to obtain a closed connected surface in S^4, then apply Proposition 2.1 and restrict to $\Sigma(B^4,S)$. □

Proposition 2.3 *Every strongly negative amphichiral knot* (K,σ) *bounds a smooth properly embedded surface* $S\subset B^4$ *which is invariant under the cone of* σ.

Proof First we fix a symmetric diagram for (K,σ), from which we will produce an equivariant unknotting sequence. Since each equivariant pair of crossing changes produces an equivariant genus-2 cobordism, this will imply that (K,σ) is equivariantly cobordant to the unknot. Then we note that the unknot bounds a smooth disk in B^4 (given by the cone of the unknot), which is invariant under the cone of σ.

For the equivariant unknotting sequence, separate K at the two fixed points of σ into two arcs, α and β. Now, for each equivariant pair of crossings between α and β, either α is the overstrand in both crossings, or β is. Hence we can perform equivariant crossing changes so that α is always the overstrand in crossings between α and β. Then we can pull α and β apart to get a knot of the form $J\#-J$, where the symmetry exchanges J and $-J$. Finally, any unknotting sequence for J produces an equivariant unknotting sequence for $J\#-J$, as desired. □

We conclude by lifting σ to the double branched cover of K.

Proposition 2.4 *Let (K,σ) be a strongly negative amphichiral knot. Then there exist exactly two lifts of σ to $\Sigma(S^3,K)$. Moreover, each such lift $\tilde{\sigma}$ has $\tilde{\sigma}^2=\tau$, where $\tau\colon \Sigma(S^3,K)\to\Sigma(S^3,K)$ is the nontrivial deck transformation action, and hence $\tilde{\sigma}$ has order 4.*

Proof The proof is essentially the same as that of Proposition 2.1. It can also be obtained by restricting the lifts in Corollary 2.2 to the boundary $\Sigma(S^3,K)$, using the surface guaranteed by Proposition 2.3. □

3 A condition on the determinant

It is implicit in the work of Goeritz [10] that the determinant of an amphichiral knot can be written as the sum of two squares (see also [24] for the converse and [9] for a partial generalization). In this section we reprove this theorem for strongly negative amphichiral knots, and show that the same condition must hold on the square root of the determinant if K is equivariantly slice.

Theorem 1.2 *Let (K,σ) be a strongly negative amphichiral knot. Then $\det(K)$ is a sum of two squares. Furthermore, if (K,σ) is equivariantly slice, then $\det(K)$ is the square of a sum of two squares.*

Before we give a proof of the theorem, we need a few lemmas.

Lemma 3.1 *Let A be an abelian group, and let $\Sigma(X,Y)$ be the double cover of a manifold X (possibly with boundary), branched over a properly embedded submanifold $Y\subset X$ with nontrivial deck transformation involution $\tau\colon\Sigma(X,Y)\to\Sigma(X,Y)$. Suppose that $H_n(X;A)=0$. Then $\tau_*(x)=-x$ for all $x\in H_n(\Sigma(X,Y);A)$.*

Proof Since $H_n(X;A)=0$, the image of the transfer homomorphism $T\colon H_n(X;A)\to H_n(\Sigma(X,Y);A)$ is 0. For any $x\in H_n(\Sigma(X,Y);A)$, we have that $x+\tau_*(x)$ is in the image of T and hence is 0. Thus $\tau_*(x)=-x$. □

Letting $(X,Y)=(S^3,K)$ in Lemma 3.1, we observe that τ_* fixes only the identity element since $H_1(\Sigma(S^3,K);A)$ has no elements of order 2.

Lemma 3.2 [4, Lemma 3] *Let K be slice with slice disk $D \subset B^4$ and A be a torsion-free abelian group. If the image of $H_1(\Sigma(S^3, K); A)$ in $H_1(\Sigma(B^4, D); A)$ has order m, then $|H_1(\Sigma(S^3, K); A)| = m^2$.*

Proof The proof is as in [4, Lemma 3], noting that since A is torsion free the universal coefficient theorem does not introduce any unwanted Tor terms. □

Lemma 3.3 *Suppose (K, σ) has an equivariant slice disk D. Then the kernel of the map*

$$i_*\colon H_1(\Sigma(S^3, K); A) \to H_1(\Sigma(B^4, D); A),$$

induced by inclusion, is invariant under the induced action of any lift $\tilde{\sigma}\colon \Sigma(S^3, K) \to \Sigma(S^3, K)$ of σ on homology.

Proof Let $x \in \ker(i_*)$ so that x is a boundary in $\Sigma(B^4, D)$. By Corollary 2.2, there is an extension of the lift $\tilde{\sigma}$ to $\Sigma(B^4, D)$. Hence $\tilde{\sigma}_*(x)$ is also a boundary, and hence contained in $\ker(i_*)$. □

Proof of Theorem 1.2 By Proposition 2.4, σ lifts to an order-4 action $\tilde{\sigma}$ on $\Sigma(S^3, K)$ with $\tilde{\sigma}^2 = \tau$. In particular, Lemma 3.1 implies that all orbits of $\tilde{\sigma}_*\colon H_1(\Sigma(S^3, K); A) \to H_1(\Sigma(S^3, K); A)$ have order 4, except the orbit consisting of the identity element. Taking coefficients A as the p–adic integers $\mathbb{Z}_p$ for some prime p, we have

$$|H_1(\Sigma(S^3, K); \mathbb{Z}_p)| \equiv 1 \pmod 4.$$

For $p \equiv 3 \pmod 4$, this implies that $|H_1(\Sigma(S^3, K); \mathbb{Z}_p)|$ is an even power of p. However, by the universal coefficient theorem, $H_1(\Sigma(S^3, K); \mathbb{Z}_p) \cong H_1(\Sigma(S^3, K); \mathbb{Z}) \otimes \mathbb{Z}_p$ and hence the prime decomposition of $|H_1(\Sigma(S^3, K); \mathbb{Z})| = \det(K)$ contains an even power of p. By the sum of two squares theorem, we then have that $\det(K)$ is the sum of two squares.

Now suppose that (K, σ) has an equivariant slice disk $D \subset B^4$. By Lemma 3.2 with p–adic coefficients, the kernel of $H_1(\Sigma(S^3, K); \mathbb{Z}_p) \to H_1(\Sigma(B^4, D); \mathbb{Z}_p)$ is a square-root order subgroup of $H_1(\Sigma(S^3, K); \mathbb{Z}_p)$, and by Lemma 3.3, this subgroup is invariant under the action of $\tilde{\sigma}_*$. In particular this subgroup must consist of the identity plus a (finite) collection of order-4 orbits, so that

$$\sqrt{|H_1(\Sigma(S^3, K); \mathbb{Z}_p)|} \equiv 1 \pmod 4.$$

As above, we then have that $\sqrt{\det(K)}$ can be written as the sum of two squares. □

4 An obstruction on Spin^c–structures

In this section we prove Theorem 1.3, giving an obstruction to an alternating strongly negative amphichiral knot bounding an equivariant slice disk D in B^4. We do so by considering Spin^c–structures on the double branched cover and applying Donaldson's theorem. This obstruction is based on the following observation:

Proposition 4.1 *Let $\rho\colon Y \to Y$ be a diffeomorphism of a closed 3–manifold Y. If ρ extends to a diffeomorphism $\rho'\colon X \to X$ of a 4–manifold X with $\partial X = Y$, then*

$$\rho^*(\mathrm{Spin}^c(X)|_Y) = \mathrm{Spin}^c(X)|_Y,$$

where $\rho^*\colon \mathrm{Spin}^c(Y) \to \mathrm{Spin}^c(Y)$ is the induced map on the Spin^c*–structures on the boundary.*

Proof Since ρ' is a diffeomorphism, $\rho^*(\mathrm{Spin}^c(X)|_Y) = (\rho')^*(\mathrm{Spin}^c(X))|_Y = \mathrm{Spin}^c(X)|_Y$. □

In order to use this proposition, take $Y = \Sigma(S^3, K)$, $X = \Sigma(B^4, D)$ and $\rho = \tilde{\sigma}\colon \Sigma(B^4, D) \to \Sigma(B^4, D)$ a lift of the strongly negative amphichiral symmetry from Corollary 2.2. In order to rule out that $\tilde{\sigma}_*(\mathrm{Spin}^c(X)|_Y) = \mathrm{Spin}^c(X)|_Y$, we will need to compute $\tilde{\sigma}^*\colon \mathrm{Spin}^c(Y) \to \mathrm{Spin}^c(Y)$ and also restrict the possible subsets $\mathrm{Spin}^c(X)|_Y \subset \mathrm{Spin}^c(Y)$ using Donaldson's theorem. Propositions 4.5 and 4.7 combined allow us to compute $\tilde{\sigma}^*\colon \mathrm{Spin}^c(Y) \to \mathrm{Spin}^c(Y)$, and Proposition 4.2 gives restrictions on $\mathrm{Spin}^c(X)|_Y \subset \mathrm{Spin}^c(Y)$. See Section 5 for an example.

We recall the following characterization of Spin^c–structures in terms of characteristic covectors which we will use throughout this section. Let X be a smooth 4–manifold which is either closed with no 2–torsion in $H_1(X)$, or constructed by attaching 2–handles to the 4–ball with ∂X a rational homology sphere. Let Q be the intersection form on X and $\mathrm{Spin}^c(X)$ be the set of Spin^c–structures of X. Then the first Chern class gives a bijection between the Spin^c–structures on X and the characteristic covectors of $H_2(X)$; see [11, Proposition 2.4.16]. More precisely,

$$\mathrm{Spin}^c(X) \cong \mathrm{Char}(H_2(X)) := \{u \in H_2(X)^* \mid u(x) \equiv Q(x,x) \pmod 2 \text{ for all } x \in H_2(X)\}.$$

In the case that $\partial X \neq \varnothing$ this identification induces a bijection

$$\mathrm{Spin}^c(\partial X) \cong \mathrm{Char}(H_2(X))/2i(H_2(X)),$$

where $i\colon H_2(X) \to H_2(X)^*$ is given by $x \mapsto Q(x, -)$ (see for example [21, Section 2.3]).

The following proposition gives restrictions on the set of Spin^c–structures on a 3–manifold which extend over a $\mathbb{Z}/2\mathbb{Z}$–homology 4–ball which it bounds. Analogous statements are discussed in [13, Section 2] and [7, Theorem 5.1].

Proposition 4.2 *Let X be a positive-definite smooth 4–manifold obtained by attaching 2–handles to the 4–ball and with ∂X a rational homology sphere Y. Suppose that Y also bounds a $\mathbb{Z}/2\mathbb{Z}$–homology 4–ball W. The inclusion map $X \to X \cup_Y -W$ induces an embedding $\iota_*\colon (H_2(X), Q) \to (\mathbb{Z}^n, \mathrm{Id})$, where Q is the intersection form of X. Choosing a basis for $H_2(X)$, ι_* is given by an $n \times n$ matrix A, and the* Spin^c*–structures on Y which extend over W are those of the form*

$$A^{\mathsf{T}}(v) \pmod{2Q} \in \mathrm{Spin}^c(Y) = \mathrm{Char}(H_2(X))/\mathrm{im}(2Q),$$

where $v \in \mathbb{Z}^n$ is any vector with all odd entries, and where elements of $\mathrm{Char}(H_2(X)) \subset \mathrm{Hom}(H_2(X), \mathbb{Z})$ are written in the dual basis.

Proof Let $Z = X \cup_Y -W$, and note that Z is positive definite (see eg [16, Proposition 7]). Hence, by Donaldson's theorem, there is an isomorphism of intersection forms $(H_2(Z)/\operatorname{Tor}, Q_Z) \cong (\mathbb{Z}^n, \mathrm{Id})$, where $n = b_2(X)$. We then have a map $\iota_*\colon (H_2(X), Q) \to (\mathbb{Z}^n, \mathrm{Id})$ induced by the inclusion $\iota\colon X \hookrightarrow Z$. Note that we may identify $\operatorname{Char}(H_2(Z))$ with $\operatorname{Spin}^c(Z)$ (since $H_1(Z)$ has no 2–torsion), and similarly $\operatorname{Char}(H_2(X))$ with $\operatorname{Spin}^c(X)$; see the discussion preceding Proposition 4.2. Applying $\operatorname{Hom}(-, \mathbb{Z})$ gives the map $\iota^*\colon H^2(Z)/\operatorname{Tor} \to H^2(X)$, which induces a map $\iota^*\colon \operatorname{Char}(H_2(Z)) \to \operatorname{Char}(H_2(X))$ on Spin^c–structures. Recall as well that the restriction $r\colon \operatorname{Spin}^c(X) \to \operatorname{Spin}^c(Y)$ is given by the quotient map

$$r\colon \operatorname{Char}(H_2(X)) \to \operatorname{Char}(H_2(X))/2i(H_2(X)),$$

where $i\colon H_2(X) \to \operatorname{Hom}(H_2(X), \mathbb{Z})$ is given by $x \mapsto Q(x, -)$. Hence the restriction map from $\operatorname{Spin}^c(Z)$ to $\operatorname{Spin}^c(Y)$ is given by $r \circ \iota^*$. We then claim that the image of $r \circ \iota^*$ is precisely the Spin^c–structures on Y which extend over W. Indeed r is surjective, so all Spin^c–structures on Y extend over X, and hence a Spin^c–structure on Y extends over W if and only if it extends over all of Z.

Combinatorially, we can compute this restriction as follows. Choose a basis for $H_2(X)$, and the dual basis for $\operatorname{Hom}(H_2(X), \mathbb{Z})$. Then ι_* is given by a matrix A, and ι^* is given by $A^\intercal$. The characteristic covectors of $H_2(Z)$ are given by vectors v in $\mathbb{Z}^n$ with all odd entries. Then the image of ι^* consists of elements of all vectors of the form

$$A^\intercal v \in \operatorname{Char}(H_2(X)) = \operatorname{Spin}^c(X),$$

written in the dual basis for $\operatorname{Hom}(H_2(X), \mathbb{Z}) \supset \operatorname{Char}(H_2(X))$. The image of $r \circ \iota^*$ then consists of these vectors modulo the column space of $2Q$. □

We now turn to computing $\tilde{\sigma}^*\colon \operatorname{Spin}^c(\Sigma(S^3, K)) \to \operatorname{Spin}^c(\Sigma(S^3, K))$. To do so, begin with a strongly negative amphichiral alternating diagram for K, and let F_+ and F_- be the pair of checkerboard surfaces with F_+ and F_- positive and negative definite, respectively. Note that F_+ and F_- are exchanged by the strongly negative amphichiral symmetry.

Definition 4.3 Take S^4 as the unit sphere in $\mathbb{R}^5$. Define $\sigma_{\mathrm{swap}}\colon S^4 \to S^4$ as the involution

$$(x_1, x_2, x_3, x_4, x_5) \mapsto (x_1, -x_2, -x_3, -x_4, -x_5).$$

On the equatorial $S^3 = \{(x_1, x_2, x_3, x_4, 0) \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$, σ_{swap} restricts to the (unique[2]) amphichiral symmetry σ with two fixed points $(\pm 1, 0, 0, 0, 0)$. Finally, note that σ_{swap} is orientation-preserving and exchanges the two hemispheres of S^4.

With respect to this involution σ_{swap}, we can push F_+ and F_- equivariantly into distinct hemispheres of S^4. By Proposition 2.1 there are two lifts, $\tilde{\sigma}_{\mathrm{swap}}$ and $\tilde{\sigma}'_{\mathrm{swap}}$, of σ_{swap} to an order-4 symmetry of

[2]Livesay [19] proved that up to conjugation there is a unique involution on S^3 with exactly two fixed points.

$\Sigma(S^4, F_+ \cup F_-)$. We have that $\tilde{\sigma}_{\mathrm{swap}} = \tilde{\sigma}'_{\mathrm{swap}} \circ \tau$, where τ is the nontrivial deck transformation involution $\tau\colon \Sigma(S^4, F_+ \cup F_-) \to \Sigma(S^4, F_+ \cup F_-)$. Using Lemma 3.1, this implies that

$$-(\tilde{\sigma}_{\mathrm{swap}})_* = (\tilde{\sigma}'_{\mathrm{swap}})_*\colon H_2(\Sigma(S^4, F_+ \cup F_-)) \to H_2(\Sigma(S^4, F_+ \cup F_-)).$$

This immediately implies the following proposition:

Proposition 4.4 *Let $\tilde{\sigma}_{\mathrm{swap}}$ and $\tilde{\sigma}'_{\mathrm{swap}}$ be the two lifts of σ_{swap} to $\Sigma(S^4, F_+ \cup F_-)$. These lifts induce maps $H_2(\Sigma(B^4, F_+)) \to H_2(-\Sigma(B^4, F_-))$ which are equal to $\pm\sigma_*\colon H_1(F_+) \to H_1(F_-)$ under the identification of $H_2(\Sigma(B^4, F_\pm))$ with $H_1(F_\pm)$ from* [12, Theorem 3].

We now use $\tilde{\sigma}_{\mathrm{swap}}$ to help us understand the action of $\tilde{\sigma}$ on Spin^c–structures.

Proposition 4.5 *Let (K, σ) be an alternating strongly negative amphichiral knot with checkerboard surfaces F_+ and F_-, and fix a lift $\tilde{\sigma}\colon \Sigma(S^3, K) \to \Sigma(S^3, K)$; see Proposition 2.4. The induced action $\tilde{\sigma}^*\colon \mathrm{Spin}^c(\Sigma(S^3, K)) \to \mathrm{Spin}^c(\Sigma(S^3, K))$ can be computed as follows. Let $\mathfrak{s} \in \mathrm{Spin}^c(\Sigma(S^3, K))$, let r, r_- and r_+ be the obvious restriction maps in the noncommutative diagram*

$$\begin{tikzcd}
\mathrm{Spin}^c(\Sigma(S^3, K)) & \mathrm{Spin}^c(\Sigma(B^4, F_+)) \arrow[l, "r"'] & \mathrm{Spin}^c(\Sigma(S^4, F_+ \cup F_-)) \arrow[l, "r_+"'] \arrow[dl, "r_-"] \\
 & \mathrm{Spin}^c(-\Sigma(B^4, F_-)) \arrow[u, "(\tilde{\sigma}^{\mathrm{res}}_{\mathrm{swap}})^*"] &
\end{tikzcd}$$

and let $\bar{\mathfrak{s}} \in \mathrm{Spin}^c(\Sigma(S^4, F_+ \cup F_-))$ be such that $r \circ r_+(\bar{\mathfrak{s}}) = \mathfrak{s}$. Then $\tilde{\sigma}^*(\mathfrak{s}) = r \circ (\tilde{\sigma}^{\mathrm{res}}_{\mathrm{swap}})^* \circ r_-(\bar{\mathfrak{s}})$, where $\tilde{\sigma}^{\mathrm{res}}_{\mathrm{swap}}\colon \mathrm{Spin}^c(-\Sigma(B^4, F_-)) \to \mathrm{Spin}^c(\Sigma(B^4, F_+))$ is the map obtained by restricting $\tilde{\sigma}_{\mathrm{swap}}$, and the lift $\tilde{\sigma}_{\mathrm{swap}}$ is chosen to agree with $\tilde{\sigma}$ on $\Sigma(S^3, K)$.

Proof By construction, $(\tilde{\sigma}_{\mathrm{swap}}|_{\Sigma(S^3,K)})^* = \tilde{\sigma}^*$. Hence the map

$$(\tilde{\sigma}_{\mathrm{swap}})^*\colon \mathrm{Spin}^c(\Sigma(S^4, F_+ \cup F_-)) \to \mathrm{Spin}^c(\Sigma(S^4, F_+ \cup F_-))$$

restricts to $\tilde{\sigma}^*\colon \mathrm{Spin}^c(\Sigma(S^3, K)) \to \mathrm{Spin}^c(\Sigma(S^3, K))$. We then compute

$$\tilde{\sigma}^*(\mathfrak{s}) = \tilde{\sigma}^* \circ r \circ r_+(\bar{\mathfrak{s}}) = r \circ r_+ \circ (\tilde{\sigma}_{\mathrm{swap}})^*(\bar{\mathfrak{s}}) = r \circ (\tilde{\sigma}^{\mathrm{res}}_{\mathrm{swap}})^* \circ r_-(\bar{\mathfrak{s}}),$$

where the final equality holds since $\tilde{\sigma}_{\mathrm{swap}}$ exchanges $\Sigma(B^4, F_+)$ and $\Sigma(B^4, F_-)$ in $\Sigma(S^4, F_+ \cup F_-)$. □

We now consider the complementary checkerboard graph $\mathcal{G}^c(F_+)$, which has a vertex v_i corresponding to each planar region of the knot diagram complementary to F_+ and an edge corresponding to each crossing in the knot diagram. Let γ_i be the simple loop in F_+ running once counterclockwise around the region corresponding to v_i. Applying the isomorphism $H_1(F_+) \cong H_2(\Sigma(B^4, F_+))$ from [12, Theorem 3], we get an element $v_i \in H_2(\Sigma(B^4, F_+))$. We call $\{v_i\}$ the *vertex generating set* of $H_2(\Sigma(B^4, F_+))$, and we declare the vertex generating set of $H_2(-\Sigma(B^4, F_+))$ to be $\{-v_i\}$.

Definition 4.6 Fix a strongly negative amphichiral alternating knot diagram, let $F_\pm$ be the positive and negative definite checkerboard surfaces and let $\mathcal{G}^c(F_\pm)$ be the corresponding complementary checkerboard

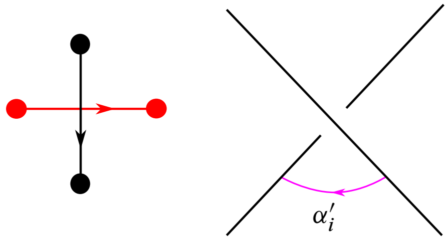


Figure 2: An oriented edge of $\mathcal{G}^c(F_+)$ in black intersecting an edge of $\mathcal{G}^c(F_-)$ in red (left). The orientation on the red edge is induced by the right-hand rule. On the right is the oriented arc α_i' induced from the oriented edge of $\mathcal{G}^c(F_+)$ in black.

graphs, embedded as dual planar graphs. The graphs $\mathcal{G}^c(F_\pm)$ are *compatibly oriented* if their edges are oriented so that intersecting dual edges satisfy the right-hand rule, as in the left of Figure 2.

Suppose $\mathcal{G}^c(F_\pm)$ are compatibly oriented, order the vertices of each of $\mathcal{G}^c(F_\pm)$ so that the strongly negative amphichiral symmetry respects the orderings and enumerate the edges of each graph so that intersecting edges have the same index; see Figure 6 for an example. We call the oriented incidence matrices $J_\pm$ for $\mathcal{G}^c(F_\pm)$ *compatible*. We use the notation J_+^* (resp. J_-^*) to denote the matrix J_+ (resp. J_-) with the last row removed. Recall that, in an oriented incidence matrix A,

$$A_{i,j} = \begin{cases} 1 & \text{if the } j^{\text{th}} \text{ edge begins at the } i^{\text{th}} \text{ vertex,} \\ -1 & \text{if the } j^{\text{th}} \text{ edge terminates at the } i^{\text{th}} \text{ vertex,} \\ 0 & \text{otherwise.} \end{cases}$$

The following proposition can be used to combinatorially compute the maps r_+ and r_- from Proposition 4.5 in terms of oriented incidence matrices; see Remark 4.9.

Proposition 4.7 *Let D be an alternating knot diagram with positive and negative definite checkerboard surfaces F_+ and F_-, respectively, and let $\mathcal{G}^c(F_\pm)$ be compatibly oriented complementary checkerboard graphs* (see Definition 4.6). *Then there is an orthonormal basis $\{e_i\}$ of $H_2(\Sigma(S^4, F_+ \cup F_-))$ in bijection with the crossings of D for which the maps $H_2(\pm\Sigma(B^4, F_\pm)) \to H_2(\Sigma(S^4, F_+ \cup F_-))$, induced by inclusion, are given by the transposes $(J_\pm)^\mathsf{T}$ of the oriented incidence matrices of $\mathcal{G}^c(F_\pm)$ with respect to the vertex generating sets for $H_2(\pm\Sigma(B^4, F_\pm))$.*

Remark 4.8 The checkerboard surfaces F_+ and F_- are always nonorientable, because they are homeomorphic and at most one checkerboard surface in any diagram can be orientable.

Proof Following [12, proof of Theorem 3], $\Sigma(B^4, F_+)$ (and similarly $\Sigma(B^4, F_-)$) can be constructed as follows. Let D_1 denote the manifold obtained by cutting open B^4 along the trace of an isotopy which pushes $\operatorname{int}(F_+)$ into $\operatorname{int}(B^4)$. The manifold D_1 is homeomorphic to B^4 and the part exposed by the cut is given by a tubular neighborhood N_+ of F_+ in $S^3 \cong \partial D_1$. Let D_2 be another copy of D_1, and let $\iota\colon N_+ \to N_+$ be the involution given by reflecting each fiber. Then

$$\Sigma(B^4, F_+) = (D_1 \cup -D_2)/(x \in N_+ \subset D_1 \sim \iota(x) \in N_+ \subset D_2).$$

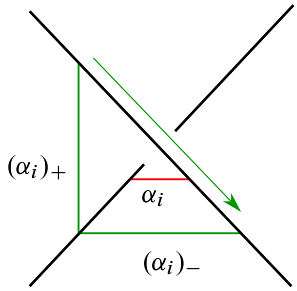


Figure 3: The arcs $(\alpha_i)_+$ and $(\alpha_i)_-$ are contained in the horizontal and vertical checkerboard surfaces, respectively. The green arrow indicates an isotopy between them in S^3. Lifting this to $\Sigma(S^3, K)$, we see that the self pairing of the sphere e_i is 1.

There is an isomorphism $\phi\colon (H_1(F_+), Q_{F_+}) \to (H_2(\Sigma(B^4, F_+)), Q_+)$, where Q_{F_+} is the Gordon–Litherland form and Q_+ is the intersection form, which is given as follows. Letting a be a 1–cycle in F_+,

$$\phi([a]) = [(\text{cone on } a \text{ in } D_1) - (\text{cone on } \iota(a) \text{ in } D_2)].$$

In their interiors, the surfaces F_+ and F_- in S^3 intersect in a collection of k arcs $\alpha_1, \dots, \alpha_k$, one for each crossing of D. The I–subbundle of N_+ over α_i is a disk $D^2_+(\alpha_i) \subset D_1$ with boundary $\tilde{\alpha}_i$, the preimage of α_i in $\Sigma(S^3, K)$. (The disk $D^2_+(\alpha_i)$ is also the trace of α_i under the isotopy pushing $\operatorname{int}(F_+)$ into $\operatorname{int}(B^4)$.) Note that $D^2_+(\alpha_i)$ is properly embedded in $\Sigma(B^4, F_+)$. Similarly, there is a disk $D^2_-(\alpha_i)$ properly embedded in $\Sigma(B^4, F_-)$, and gluing these disks along $\tilde{\alpha}_i$ gives a sphere e_i in $\Sigma(S^4, F_+ \cup F_-)$.

Note that $e_1, \dots, e_k$ are in correspondence with the edges of $\mathcal{G}^c(F_+)$ (and $\mathcal{G}^c(F_-)$). Furthermore, the orientation on an edge E_i in $\mathcal{G}^c(F_+)$ induces an orientation on the corresponding e_i as follows. First, orient the arc α_i going into the page of the knot diagram (away from the reader). Next, push the interior of α_i into the region corresponding to the terminal vertex of E_i and then out of the page of the diagram (toward the reader) so that it is disjoint from $F_+ \cup F_-$. Call the resulting arc α_i'; see Figure 2. Recall that $\Sigma(B^4, F_+) = D_1 \cup -D_2$ as an oriented manifold. Then the orientation of $\alpha_i' \subset D_1$ determines an orientation on the union of $\alpha_i' \subset D_1$ with $-\alpha_i' \subset -D_2$, which is locally isotopic within $\Sigma(S^3, K)$ to $\tilde{\alpha}_i$. This orientation on $\tilde{\alpha}_i$ then determines an orientation on $D^2_+(\alpha_i)$ as its oriented boundary, and this orientation on $D^2_+(\alpha_i)$ extends to an orientation on $e_i = D^2_+(\alpha_i) \cup D^2_-(\alpha_i)$.

We now show that $\{e_1, \dots, e_k\}$ is an orthonormal basis for $H_2(\Sigma(S^4, F_+ \cup F_-))$. Note that

$$b_2(\Sigma(S^4, F_+ \cup F_-)) = b_2(\Sigma(B^4, F_+)) + b_2(\Sigma(B^4, F_-)),$$

since $\Sigma(S^3, K)$ is a rational homology sphere. However, $b_2(\Sigma(B^4, F_\pm)) = n_\pm - 1$, where $n_\pm$ is the number of vertices of $\mathcal{G}^c(F_\pm)$. From the Euler characteristic of the sphere of the knot diagram, we get $2 = n_+ - k + n_-$ since $\mathcal{G}^c(F_+)$ and $\mathcal{G}^c(F_-)$ are dual graphs. Hence $b_2(\Sigma(S^4, F_+ \cup F_-)) = k$. Thus it suffices to show that $e_1, \dots, e_k$ are orthonormal. Observe that e_i and e_j are disjoint for $i \neq j$, so it is enough to show that $e_i \cdot e_i = 1$. Consider the arcs $(\alpha_i)_\pm$ shown in Figure 3, where $(\alpha_i)_\pm \subset F_\pm$ and $(\alpha_i)_+$ intersects $(\alpha_i)_-$

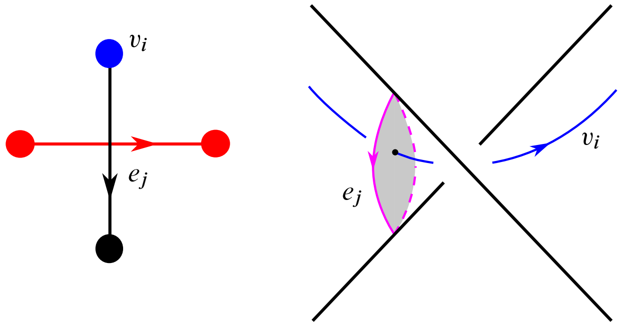


Figure 4: If $v_i \in \mathcal{G}^c(F_+)$ is the starting endpoint of an edge corresponding to e_j, then $e_j \cdot v_i = 1$. The magenta loop is the boundary of the gray disk $e_j \cap N_+$, and is oriented so that the arc coming out of the page is isotopic (keeping the endpoints on K) to α_j' (see Figure 2) in the complement of $F_+ \cup F_-$.

at a single point. Observe that the preimages $(\tilde{\alpha}_i)_\pm \subset \Sigma(B^4, F_\pm)$ of $(\alpha_i)_\pm$ bound disks $D^2_\pm(\alpha_i)'$ parallel to $D^2_\pm(\alpha_i)$ in $\Sigma(B^4, F_\pm)$. There is an isotopy in S^3 between $(\alpha_i)_+$ and $(\alpha_i)_-$ intersecting α_i in a single point, which induces an isotopy between $(\tilde{\alpha}_i)_+$ and $(\tilde{\alpha}_i)_-$. Gluing $D^2_+(\alpha_i)'$ to $D^2_-(\alpha_i)'$ along the (image of the) isotopy in $\Sigma(S^3, K)$ defines a push-off of e_i which has a single positive transverse intersection with e_i.

Recall that an element $v_i \in H_2(\Sigma(B^4, F_+))$ of the vertex generating set is represented by a sphere which intersects $N_+ \subset \Sigma(B^4, F_+)$ in a loop $\gamma_i \subset F_+$. By construction, $e_j \cap \Sigma(B^4, F_+)$ is the disk $D^2_+(\alpha_j)$ contained in $N_+ \subset \Sigma(B^4, F_+)$. Hence $v_i \cdot e_j$ can be computed locally in N_+. Diagrammatically (see Figure 4), we draw $\partial D_1 = S^3$ and think of N_+ as a neighborhood of $F_+ \subset S^3$. Specifically, $v_i \cdot e_j = 0$ if the edge corresponding to e_j and v_i are not incident, $v_i \cdot e_j = 1$ if the edge corresponding to e_j begins at v_i, and $v_i \cdot e_j = -1$ if the edge corresponding to e_j terminates at v_i. A similar argument applies to the vertex generating set of $H_2(-\Sigma(B^4, F_-))$. □

Remark 4.9 Proposition 4.7 combinatorially determines the maps

$$r_\pm \colon \mathrm{Spin}^c(\Sigma(S^4, F_+ \cup F_-)) \to \mathrm{Spin}^c(\pm\Sigma(B^4, F_\pm))$$

from Proposition 4.5. Specifically, the maps $r_\pm$ are given by taking the duals of

$$H_2(\pm\Sigma(B^4, F_\pm)) \to H_2(\Sigma(S^4, F_+ \cup F_-)),$$

then restricting to characteristic vectors.

We conclude the section with a proof of Theorem 1.3 from the introduction:

Proof of Theorem 1.3 Let $Y = \Sigma(S^3, K)$ and $X_\pm = \Sigma(B^4, F_\pm)$. We identify each of $H_2(X_\pm)$ with the $\mathbb{Z}$–span of $\mathrm{Vert}(\mathcal{G}^c(F_\pm)) \backslash \{v_\pm\}$, where $\{v_+, v_-\}$ is the pair of σ–invariant vertices removed when defining $J^*_\pm$. Note that $X_\pm$ can be constructed by attaching 2–handles to the 4–ball (see for example the proof of Lemma 3.6 in [22]). Hence, using the dual basis for $H_2(X_\pm)^*$, we may identify

$$\mathrm{Spin}^c(X_\pm) \cong \mathrm{Char}(\mathbb{Z}^n, A_\pm) \quad \text{and} \quad \mathrm{Spin}^c(Y) \cong \mathrm{Char}(\mathbb{Z}^n, A_+)/\mathrm{im}(2A_+);$$

see the discussion before Proposition 4.2. With respect to these choices of dual bases, we may choose a lift $\tilde{\sigma}$ of σ to Y so that $\tilde{\sigma}^*_{\mathrm{swap}}\colon H_2(-X_-)^* \to H_2(X_+)^*$ is the identity matrix by Proposition 4.4; this determines the map on Spin^c–structures. Since Y is a rational homology sphere, $b_2(\Sigma(S^4, F_+ \cup F_-)) = b_2(\Sigma(B^4, F_+)) + b_2(\Sigma(B^4, F_-)) = n+n$. Using the orthonormal basis for $H_2(\Sigma(S^4, F_+ \cup F_-)) \cong \mathbb{Z}^{2n}$ from Proposition 4.7, we may identify

$$\mathrm{Spin}^c(\Sigma(S^4, F_+ \cup F_-)) \cong \{v \in \mathbb{Z}^{2n} \mid v \equiv (1,1,\dots,1)^{\mathsf{T}} \pmod 2\}.$$

By Proposition 4.7 (see also Remark 4.9), the maps $r_\pm$ in Proposition 4.5 are given by $J^*_\pm$. Proposition 4.5 then shows that the map $\tilde{\sigma}^*\colon \mathrm{Spin}^c(Y) \to \mathrm{Spin}^c(Y)$ is determined by

$$\tilde{\sigma}^*[J^*_+ v] = [J^*_- v] \quad \text{for all } v \in \mathbb{Z}^{2n} \text{ with } v \equiv (1,1,\dots,1)^{\mathsf{T}} \pmod 2.$$

Finally, let D be an equivariant slice disk for K. By Proposition 4.2, the set of Spin^c–structures of Y which extend over $\Sigma(B^4, D)$ is given by

$$S = \{[u] \in \mathrm{Spin}^c(Y) \mid u = A^{\mathsf{T}} v \text{ for some } v \in \mathbb{Z}^n \text{ with } v \equiv (1,1,\dots,1)^{\mathsf{T}} \pmod 2\},$$

and by Corollary 2.2 there is a lift $\Sigma(B^4, D) \to \Sigma(B^4, D)$ which restricts to the lift $\tilde{\sigma}$ on Y. Hence, by Proposition 4.1, S is $\tilde{\sigma}^*$–invariant. □

5 An alternating slice strongly negative amphichiral example

In this section we give an example of a strongly negative amphichiral knot which Theorem 1.3 shows is not equivariantly slice.

Example 5.1 Consider the slice knot $K = 12a_{1105}$ along with the strongly negative amphichiral alternating diagram shown in Figure 5. Theorem 1.3 obstructs K from being equivariantly slice. Note that Theorem 1.2 does not provide an obstruction since $\det(K) = 17^2$. Let F_+ (resp. F_-) be the positive (resp. negative) definite checkerboard surface for the knot diagram in Figure 5. In Figure 6 we draw corresponding compatibly oriented complementary checkerboard graphs $\mathcal{G}^c(F_\pm)$. The edges in each

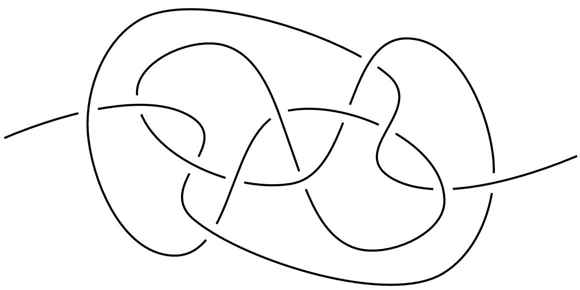


Figure 5: A strongly negative amphichiral symmetry on $12a_{1105}$. The symmetry is π–rotation within the plane of the diagram followed by a reflection across the plane of the diagram.

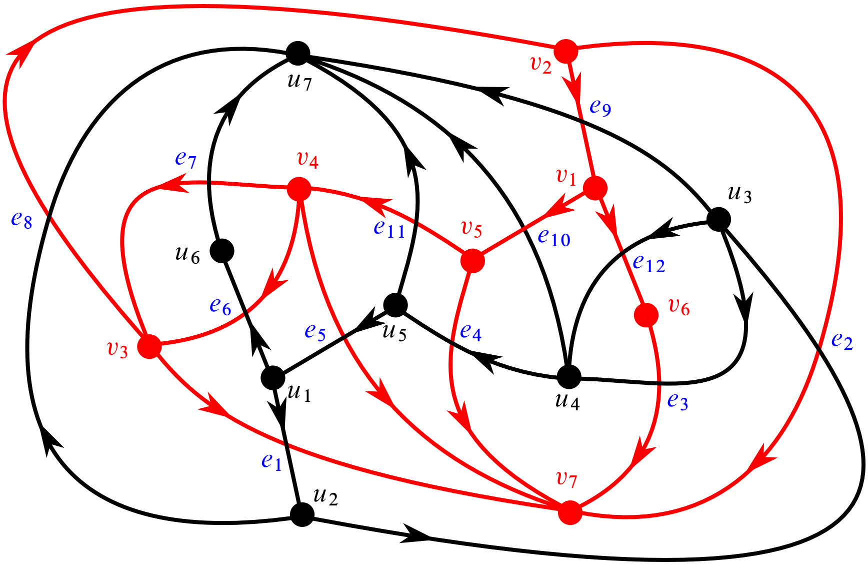


Figure 6: The pair of complementary checkerboard graphs of the alternating diagram for $12a_{1105}$ in Figure 5. They are exchanged by the strongly negative amphichiral symmetry. $\mathcal{G}^c(F_+)$ is black and $\mathcal{G}^c(F_-)$ is red. The $\{e_i\}$ correspond to crossings in the knot diagram.

graph are enumerated by the crossings e_i shown in Figure 6. Using u_7 and v_7 for the last row of the oriented incidence matrices $J_\pm$ (which we remove), we have

$$J_+^* = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$J_-^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

From these we can compute the Goeritz matrix for F_+:

$$A_+ = J_+^*(J_+^*)^\mathsf{T} = \begin{bmatrix} 3 & -1 & 0 & 0 & -1 & -1 \\ -1 & 3 & -1 & 0 & 0 & 0 \\ 0 & -1 & 4 & -2 & 0 & 0 \\ 0 & 0 & -2 & 4 & -1 & 0 \\ -1 & 0 & 0 & -1 & 3 & 0 \\ -1 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

We now combinatorially enumerate all possible lattice embeddings $A\colon (\mathbb{Z}^6, A_+) \to (\mathbb{Z}^6, \mathrm{Id})$, up to automorphisms of $\mathbb{Z}^6$. Using a computer program[3] we enumerate integer matrices A satisfying $A^{\mathsf{T}}A = A_+$, up to permutations and sign changes of the rows of A. We find two possibilities for A, which we denote by A_1 and A_2; their transposes are

$$A_1^{\mathsf{T}} = \begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A_2^{\mathsf{T}} = \begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ -1 & 0 & -1 & -1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Neither matrix satisfies the $\tilde{\sigma}^*$–invariance condition in Theorem 1.3. We will show this for the matrix A_1; the computation for A_2 is similar. For A_1, we compute that the set

$$S = \{[u] \in \mathrm{Spin}^c(Y) \mid u = A_1^{\mathsf{T}} v \text{ for some } v \in \mathbb{Z}^n \text{ with } v \equiv (1,1,\dots,1)^{\mathsf{T}} \pmod 2\}$$

consists of the 17 classes represented by the following vectors:

$$\begin{bmatrix} 1 \\ 1 \\ -2 \\ -2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ 2 \\ 0 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ -2 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -4 \\ 4 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 4 \\ -2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \\ 4 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -2 \\ 2 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 6 \\ -4 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 2 \\ -3 \\ 2 \end{bmatrix},$$
$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ -2 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -6 \\ 4 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 2 \\ -2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 2 \\ -4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -4 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 4 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 0 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ -2 \\ 0 \\ 1 \\ 2 \end{bmatrix}.$$

We will show that this collection S of Spin^c–structures on $\Sigma(S^3, K)$ is not $\tilde{\sigma}^*$–invariant. Specifically, we will show that the Spin^c–structure represented by the second vector $\mathfrak{s} = (3,-3,2,0,-1,-2)^{\mathsf{T}}$ is mapped by $\tilde{\sigma}^*$ to a Spin^c–structure not contained in S.

Consider the vector

$$\tilde{\mathfrak{s}} = (7,3,3,3,1,-3,-5,1,1,1,1,1)^{\mathsf{T}} \in \mathbb{Z}^{12}.$$

Multiplying, we see that $J_+^*(\tilde{\mathfrak{s}}) = \mathfrak{s}$ and $J_-^*(\tilde{\mathfrak{s}}) = (1,3,16,-8,3,2)^{\mathsf{T}}$. A straightforward linear algebra computation shows that $(1,3,16,-8,3,2)^{\mathsf{T}}$ is not equivalent modulo $2A_+$ to any of the 17 vectors in S. Hence $\tilde{\sigma}^*[J_+^*(\tilde{\mathfrak{s}})] = [J_-^*(\tilde{\mathfrak{s}})]$ is not in S. Along with a similar computation for A_2, this implies that K is not equivariantly slice, by Theorem 1.3.

[3] The equation $A^{\mathsf{T}}A = A_+$ implies that each column of A has bounded norm, so there are finitely many possibilities to check for A.

6 Heegaard Floer correction terms

In this section we give a necessary condition on the Heegaard Floer correction terms $d(\Sigma(S^3, K), \mathfrak{s})$, also known as d–invariants, for a knot to be strongly negative amphichiral. In the case of periodic knots, a similar type of condition was proved by Jabuka and Naik in [17]. As in the case of periodic knots, we first need invariance of the d–invariants.

Lemma 6.1 *Let Y be a rational homology 3–sphere with $\mathfrak{s} \in \mathrm{Spin}^c(Y)$ and $\sigma\colon Y \to Y$ an orientation-reversing diffeomorphism. Then*

$$d(Y, \sigma^*(\mathfrak{s})) = -d(Y, \mathfrak{s}).$$

Proof This follows directly from the diffeomorphism invariance of Heegaard Floer homology. □

Along with the following lemma, this implies our final theorem below.

Lemma 6.2 *For any knot $K \subset S^3$, the deck transformation involution τ of the double branched cover $\Sigma(S^3, K)$ acts on the set of Spin^c–structures by conjugation.*

Proof The first Chern class $c_1\colon \mathrm{Spin}^c(\Sigma(S^3, K)) \to H^2(\Sigma(S^3, K); \mathbb{Z})$ is an isomorphism, since $\Sigma(S^3, K)$ is a $\mathbb{Z}/2\mathbb{Z}$ homology sphere, and by Poincaré duality we also have an isomorphism

$$H^2(\Sigma(S^3, K); \mathbb{Z}) \cong H_1(\Sigma(S^3, K); \mathbb{Z}).$$

By Lemma 3.1, τ acts as the negative of the identity on $H_1(\Sigma(S^3, K); \mathbb{Z})$, which then induces conjugation on the set of Spin^c–structures under these natural isomorphisms. □

Theorem 1.4 *Let (K, σ) be a strongly negative amphichiral knot and let $\tilde{\sigma}$ be a lift of σ to $\Sigma(S^3, K)$* (see Proposition 2.1). *Then the orbits of the d–invariants of $\Sigma(S^3, K)$ under the action of $\tilde{\sigma}$ satisfy:*

- *There is exactly one orbit $\{\mathfrak{s}_0\}$ of order 1. Moreover, $d(\Sigma(S^3, K), \mathfrak{s}_0) = 0$.*
- *Other orbits $\{\mathfrak{s}, \tilde{\sigma}(\mathfrak{s}), \tilde{\sigma}^2(\mathfrak{s}), \tilde{\sigma}^3(\mathfrak{s})\}$ have order 4, and $d(\Sigma(S^3, K), \tilde{\sigma}^i(\mathfrak{s})) = (-1)^i r$ for some $r \in \mathbb{Q}$.*

Proof Since $\tilde{\sigma}$ has order 4, the $\tilde{\sigma}^*$–orbits of the Spin^c–structures will have order 1, 2 or 4. Let $\tau = \tilde{\sigma}^2$ be the deck transformation action on $\Sigma(S^3, K)$, and note that τ^* acts on the set of Spin^c–structures by conjugation by Lemma 6.2. Hence, if a Spin^c–structure is not fixed by conjugation, then it will have a $\tilde{\sigma}^*$–orbit of length 4. On the other hand, since $\Sigma(S^3, K)$ is a $\mathbb{Z}/2\mathbb{Z}$–homology sphere, there is a unique Spin^c–structure $\mathfrak{s}_0$ fixed by conjugation. Furthermore, since $|H_1(\Sigma(S^3, K)|$ is odd there are an odd number of Spin^c–structures, and hence $\mathfrak{s}_0$ has a $\tilde{\sigma}^*$–orbit of length 1. □

Example 6.3 The d–invariants of $\Sigma(S^3, 6_1)$, appropriately oriented, are

$$-\tfrac{4}{9}, -\tfrac{4}{9}, 0, 0, 0, \tfrac{2}{9}, \tfrac{2}{9}, \tfrac{8}{9}, \tfrac{8}{9}.$$

Since these are not of the form required by Theorem 1.4, 6_1 is not strongly negative amphichiral. We compare this to the strongly negative amphichiral knot 6_3, for which $\Sigma(S^3, 6_3)$ has d–invariants

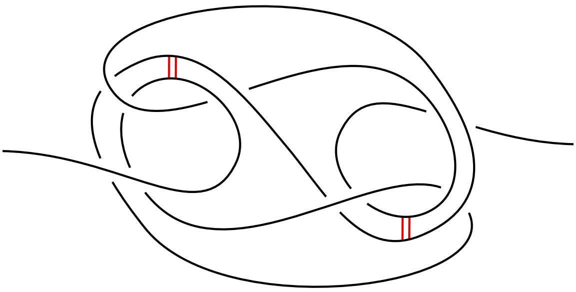
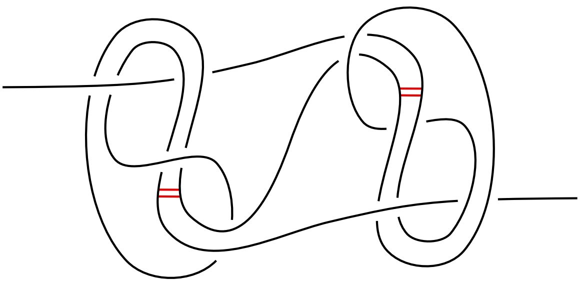
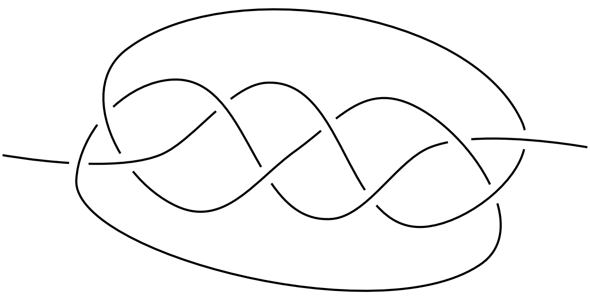
$$0, \tfrac{8}{13}, -\tfrac{8}{13}, \tfrac{8}{13}, -\tfrac{8}{13}, \tfrac{6}{13}, -\tfrac{6}{13}, \tfrac{6}{13}, -\tfrac{6}{13}, \tfrac{2}{13}, -\tfrac{2}{13}, \tfrac{2}{13}, -\tfrac{2}{13}.$$

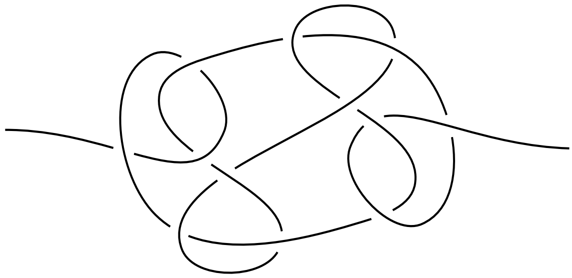
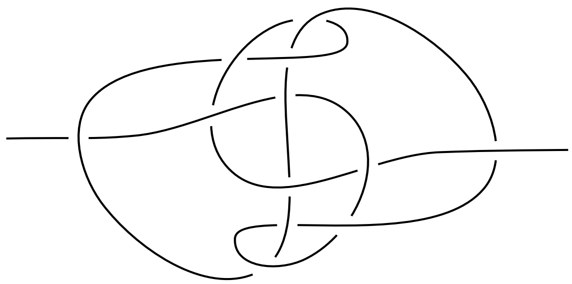
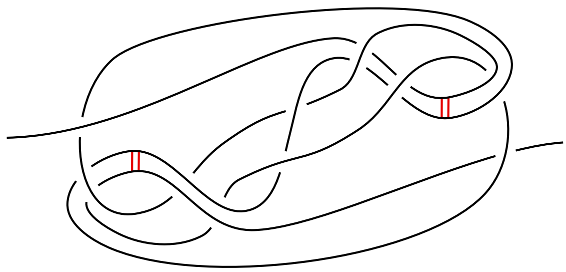
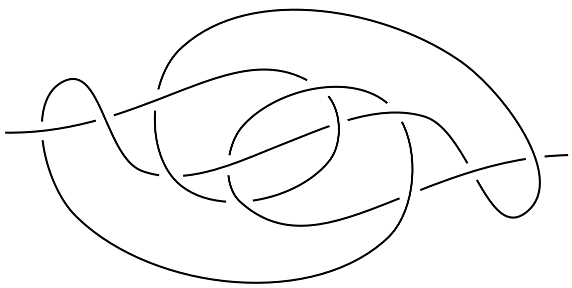
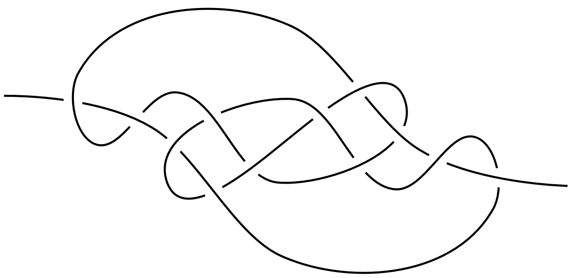
7 A table of slice strongly negative amphichiral prime knots with 12 or fewer crossings

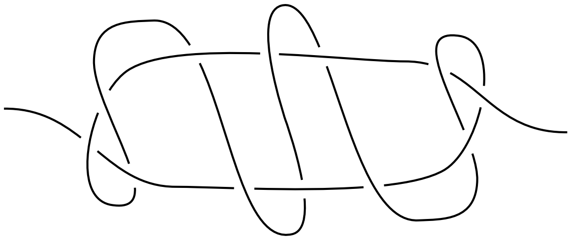
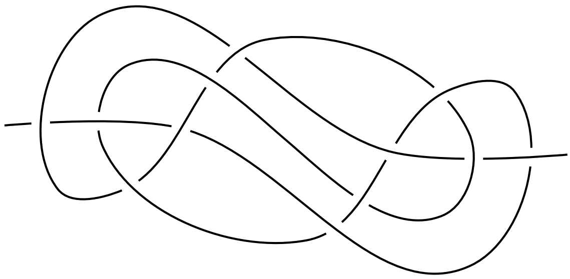
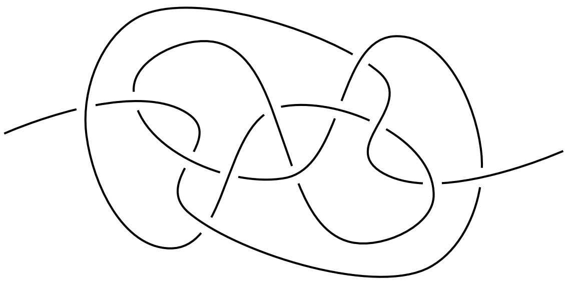
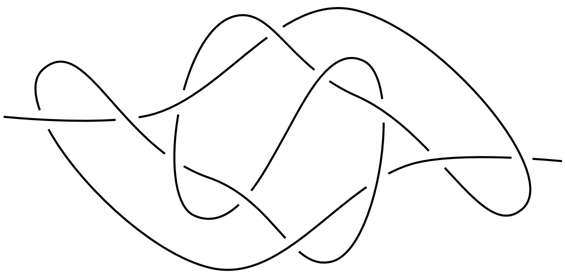
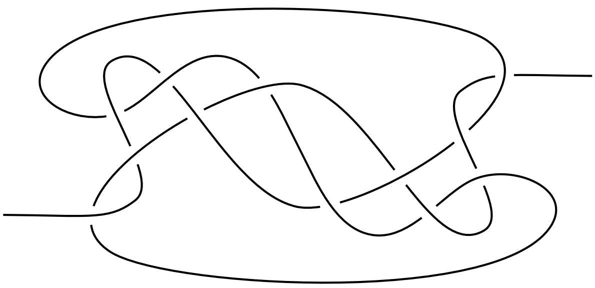
We conclude with a table of all slice strongly negative amphichiral prime knots with 12 or fewer crossings. These are categorized as follows:

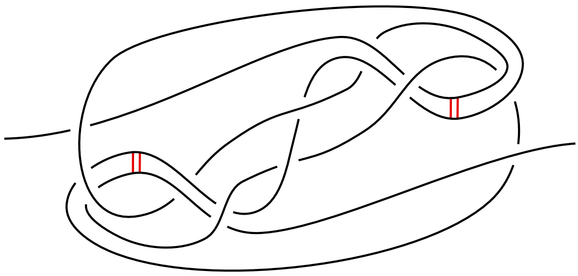
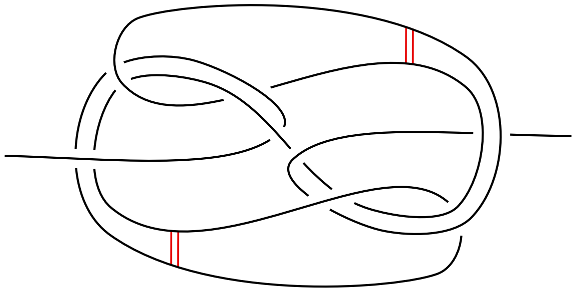
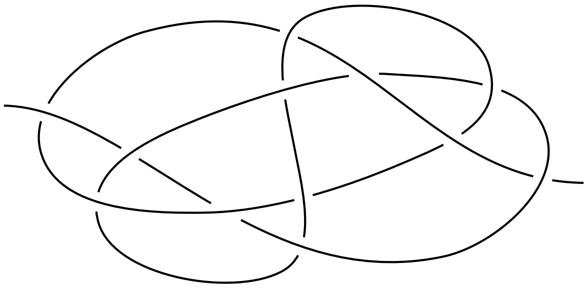
- (Rib) Knots for which we have found an equivariant ribbon diagram. We indicate this with a pair of equivariant bands (in red), which reduce the knot to a 3–component unlink.
- (Det) Knots for which Theorem 1.2 obstructs an equivariant slice disk.
- (Spin^c) Knots for which the obstruction from Theorem 1.2 fails, but Theorem 1.3 obstructs an equivariant slice disk.
- (Unk) Knots for which we were unable to find or obstruct an equivariant slice disk.

We also include the knot determinant and whether the knot is equivariantly slice.

name	diagram	eq. slice	category	det
8_9		yes	(Rib)	5^2
10_{99}		yes	(Rib)	9^2
10_{123}		no	(Det)	11^2

name	diagram	eq. slice	category	det
$12a_{435}$		no	(Det)	15^2
$12a_{458}$		unknown	(Unk)	17^2
$12a_{819}$		yes	(Rib)	13^2
$12a_{887}$		unknown	(Unk)	17^2
$12a_{990}$		no	(Det)	15^2

name	diagram	eq. slice	category	det
$12a_{477}$		unknown	(Unk)	13^2
$12a_{1019}$		no	(Det)	19^2
$12a_{1105}$		no	(Spin^c)	17^2
$12a_{1202}$		no	(Spin^c)	13^2
$12a_{1225}$		no	(Det)	15^2

name	diagram	eq. slice	category	det
$12a_{1269}$		yes	(Rib)	13^2
$12n_{462}$		yes	(Rib)	5^2
$12n_{706}$		no	(Det)	7^2

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ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 24 Issue 2 (pages 595–1223) 2024

Comparing combinatorial models of moduli space and their compactifications 595

DANIELA EGAS SANTANDER and ALEXANDER KUPERS

Towards a higher-dimensional construction of stable/unstable Lagrangian laminations 655

SANGJIN LEE

A strong Haken theorem 717

MARTIN SCHARLEMANN

Right-angled Artin subgroups of right-angled Coxeter and Artin groups 755

PALLAVI DANI and IVAN LEVCOVITZ

Filling braided links with trisected surfaces 803

JEFFREY MEIER

Equivariantly slicing strongly negative amphichiral knots 897

KEEGAN BOYLE and AHMAD ISSA

Computing the Morava K–theory of real Grassmannians using chromatic fixed point theory 919

NICHOLAS J KUHN and CHRISTOPHER J R LLOYD

Slope gap distributions of Veech surfaces 951

LUIS KUMANDURI, ANTHONY SANCHEZ and JANE WANG

Embedding calculus for surfaces 981

MANUEL KRANNICH and ALEXANDER KUPERS

Vietoris–Rips persistent homology, injective metric spaces, and the filling radius 1019

SUNHYUK LIM, FACUNDO MÉMOLI and OSMAN BERAT OKUTAN

Slopes and concordance of links 1101

ALEX DEGTYAREV, VINCENT FLORENS and ANA G LECUONA

Cohomological and geometric invariants of simple complexes of groups 1121

NANSEN PETROSYAN and TOMASZ PRYTUŁA

On the decategorification of some higher actions in Heegaard Floer homology 1157

ANDREW MANION

A simplicial version of the 2–dimensional Fulton–MacPherson operad 1183

NATHANIEL BOTTMAN

Intrinsically knotted graphs with linklessly embeddable simple minors 1203

THOMAS W MATTMAN, RAMIN NAIMI, ANDREI PAVELESCU and ELENA PAVELESCU