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Computing the Morava K–theory of real Grassmannians using chromatic fixed point theory

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We study $K(n)^*(\mathrm{Gr}_d(\mathbb{R}^m))$, the 2–local Morava K–theories of the real Grassmannians, about which very little has been previously computed. We conjecture that the Atiyah–Hirzebruch spectral sequences computing these all collapse after the first possible nonzero differential $d_{2^{n+1}-1}$, and give much evidence that this is the case.

We use a novel method to show that higher differentials can't occur: we get a lower bound on the size of $K(n)^*(\mathrm{Gr}_d(\mathbb{R}^m))$ by constructing a C_4–action on our Grassmannians and then applying the chromatic fixed point theory of the authors' previous paper. In essence, we bound the size of $K(n)^*(\mathrm{Gr}_d(\mathbb{R}^m))$ by computing $K(n-1)^*(\mathrm{Gr}_d(\mathbb{R}^m)^{C_4})$.

Meanwhile, the size of $E_{2^{n+1}}$ is given by Q_n–homology, where Q_n is Milnor's n^{th} primitive mod 2 cohomology operation. Whenever we are able to calculate this Q_n–homology, we have found that the size of $E_{2^{n+1}}$ agrees with our lower bound for the size of $K(n)^*(\mathrm{Gr}_d(\mathbb{R}^m))$. We have two general families where we prove this: $m \leq 2^{n+1}$ and all d, and $d = 2$ and all m and n. Computer calculations have allowed us to check many other examples with larger values of d.

55M35, 55N20; 55P91, 57S17

1 Introduction

Let $\mathrm{Gr}_d(\mathbb{R}^m)$ be the real Grassmannian of k–planes in $\mathbb{R}^m$, a much studied compact manifold of dimension $d(m-d)$ admitting the structure of a CW complex with $\binom{m}{d}$ "Schubert cells".

Much is known about the ordinary cohomology of these spaces:

(1) $H^*(\mathrm{Gr}_d(\mathbb{R}^m); \mathbb{Z}/2)$ is generated by Stiefel–Whitney classes satisfying standard relations. It has total dimension $\binom{m}{d}$.

(2) $H^*(\mathrm{Gr}_d(\mathbb{R}^m); \mathbb{Q})$ is generated by Pontryagin classes, along with, in some cases, an odd-dimensional class. For fixed d, and $\epsilon = 0$ or 1, the total dimension of $H^*(\mathrm{Gr}_d(\mathbb{R}^{2-\epsilon+2l}); \mathbb{Q})$ is polynomial of degree $\lfloor d/2 \rfloor$ as a function of $l \geq 0$.

(3) If m is even, then $\mathrm{Gr}_d(\mathbb{R}^m)$ is oriented. Furthermore, the inclusion $\mathrm{Gr}_d(\mathbb{R}^{m-1}) \hookrightarrow \mathrm{Gr}_d(\mathbb{R}^m)$ induces an epimorphism in rational cohomology.

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(4) Nontrivial torsion in $H^*(\mathrm{Gr}_d(\mathbb{R}^m);\mathbb{Z})$ has order 2. The mod 2 Bockstein spectral sequence (BSS) collapses after the first differential. Equivalently, the mod 2 Adams spectral sequence (ASS) converging to $H^*(\mathrm{Gr}_d(\mathbb{R}^m);\mathbb{Z})$ collapses at E_2.

Much less is known about other cohomology theories applied to these Grassmannians. In this paper, we study $K(n)^*(\mathrm{Gr}_d(\mathbb{R}^m))$ for $n \geq 1$. Here $K(n)^*(X)$ denotes the 2–local n^{th} Morava K–theory of a space X, a graded vector space over the graded field $K(n)_* = \mathbb{Z}/2[v_n^{\pm}]$ with $|v_n| = 2^{n+1}-2$. We let $k(n)$ denote the connective cover of $K(n)$: $k(n)_* = \mathbb{Z}/2[v_n]$.

Viewing $H\mathbb{Q}$ as $K(0)$ and $H\mathbb{Z}$ as $k(0)$, our discovery is that analogues of statements (2)–(4) above appear to hold for all n, with the Atiyah–Hirzebruch spectral sequence (AHSS) replacing the Bockstein spectral sequence in statement (4). Furthermore, the analogue of statement (1) holds through a much bigger range than one would expect from dimension considerations.

In the next two subsections, we describe our main results.

1.1 Results proved using chromatic fixed point theory

Given a finite complex X and $n \geq 0$, we let $k_n(X) = \dim_{K(n)_*} K(n)^*(X)$.

Theorem 1.1 *If* $m \leq 2^{n+1}$, *then* $k_n(\mathrm{Gr}_d(\mathbb{R}^m)) = \binom{m}{d}$. *Thus, in this range, the AHSS converging to* $K(n)^*(\mathrm{Gr}_d(\mathbb{R}^m))$ *collapses at* E_2.

We note that this collapsing range is surprisingly large, as dimension considerations just imply collapsing if $d(m-d) < 2^{n+1}$.

For larger m, we have the following lower bound.

Theorem 1.2 *Let* $m = 2^{n+1} - \epsilon + 2l$ *with* $\epsilon = 0$ *or* 1, *and* $l \geq 0$. *Then*

$$k_n(\mathrm{Gr}_d(\mathbb{R}^m)) \geq \sum_{i=0}^{\lfloor d/2 \rfloor} \binom{2^{n+1}-\epsilon}{d-2i}\binom{l}{i}.$$

Conjecture 1.3 *Equality always holds in this last theorem.*

The biggest novelty of this paper is our method for proving Theorems 1.1 and 1.2: we make use of chromatic fixed point theory to prove these nonequivariant results.

The blue shift theorem of Barthel, Hausmann, Naumann, Nikolaus, Noel and Stapleton [2] says that if C is a finite cyclic p–group and X is a finite C–CW complex, then

$$\widetilde{K}(n)^*(X) = 0 \implies \widetilde{K}(n-1)^*(X^C) = 0;$$

see also Balderrama and the first author [1]. In [8], we upgraded this. Specialized to cyclic groups, [8, Theorem 2.17] says the following.

Theorem 1.4 *If C is a finite cyclic p–group, and X is a finite C–CW complex, then*

$$k_n(X) \geq k_{n-1}(X^C).$$

Note that, in these statements, $K(n)_*$ means Morava K–theory at the prime p.

As $\binom{m}{d}$ is an evident upper bound for $k_n(\mathrm{Gr}_d(\mathbb{R}^m))$, to prove Theorem 1.1, it suffices to show that $k_n(\mathrm{Gr}_d(\mathbb{R}^m)) \geq \binom{m}{d}$ in the stated range. Using Theorem 1.4, we will show this by induction on n using a C_2–action on $\mathrm{Gr}_d(\mathbb{R}^m)$ induced by an m–dimensional real representation of $C = C_2$.

We will similarly prove Theorem 1.2 for $n \geq 1$ by using a C_4–action on $\mathrm{Gr}_d(\mathbb{R}^m)$ induced by an m–dimensional real representation of $C = C_4$.

In both cases, it will be quite easy to compute $k_{n-1}(\mathrm{Gr}_d(\mathbb{R}^m)^C)$.

Details of this will be in Section 2.

1.2 Results about the Q_n–homology of the Grassmannians

Conjecture 1.3 follows from a conjectural calculation that only involves $H^*(\mathrm{Gr}_d(\mathbb{R}^m); \mathbb{Z}/2)$, viewed as a module over the Steenrod algebra.

Let Q_n for $n = 0, 1, 2, \ldots$ be the Milnor primitives: the elements in the mod 2 Steenrod algebra recursively defined by $Q_0 = \mathrm{Sq}^1$, and $Q_n = [Q_{n-1}, \mathrm{Sq}^{2^n}]$. These satisfy $Q_n^2 = 0$, and we let $k_{Q_n}(X)$ denote the total dimension of the *Q_n–homology of X*,

$$H^*(X; Q_n) = \frac{Z^*(X; Q_n)}{B^*(X; Q_n)},$$

where

$$\begin{aligned} Z^*(X; Q_n) &= \ker\{Q_n \colon H^*(X; \mathbb{Z}/2) \to H^{*+2^{n+1}-1}(X; \mathbb{Z}/2)\}, \\ B^*(X; Q_n) &= \mathrm{im}\{Q_n \colon H^{*-2^{n+1}+1}(X; \mathbb{Z}/2) \to H^*(X; \mathbb{Z}/2)\}. \end{aligned}$$

As will be reviewed in Section 3.1, the first differential in the AHSS converging to $K(n)^*(X)$ is $d_{2^{n+1}-1}$, with formula

$$d_{2^{n+1}-1}(x) = Q_n(x)v_n$$

for all $x \in E_2^{*,0}(X) = H^*(X; \mathbb{Z}/2)$. This makes it not hard to check the next lemma.

Lemma 1.5 *If X is a finite complex, $k_{Q_n}(X) \geq k_n(X)$ is always true, and the following are equivalent:*

(a) $k_{Q_n}(X) = k_n(X)$;

(b) *the AHSS, when $n \geq 1$, or the BSS, when $n = 0$, computing $K(n)^*(X)$ collapses at $E_{2^{n+1}}$;*

(c) *the ASS computing $k(n)^*(X)$ collapses at E_2.*

We apply this to our situation. First, Theorem 1.1 has the following nontrivial algebraic consequence.

Corollary 1.6 *If $m \leq 2^{n+1}$, then Q_n acts trivially on $H^*(\mathrm{Gr}_d(\mathbb{R}^m);\mathbb{Z}/2)$.*

For an algebraic proof of this result using the methods of Section 3.5, see the second author's thesis [10, page 75].

For $m > 2^{n+1}$, we believe the following is true.

Conjecture 1.7 *Let $m = 2^{n+1} - \epsilon + 2l$ with $\epsilon = 0$ or 1, and $l \geq 0$. Then*

$$k_{Q_n}(\mathrm{Gr}_d(\mathbb{R}^m)) = \sum_{i=0}^{\lfloor d/2 \rfloor} \binom{2^{n+1}-\epsilon}{d-2i}\binom{l}{i}.$$

Comparison with Theorem 1.2 shows that when Conjecture 1.7 is true, one can conclude

- $k_{Q_n}(\mathrm{Gr}_d(\mathbb{R}^m)) = k_n(\mathrm{Gr}_d(\mathbb{R}^m))$, and Conjecture 1.3 is true;
- the AHSS computing $K(n)^*(\mathrm{Gr}_d(\mathbb{R}^m))$ collapses at $E_{2^{n+1}}$;
- the ASS computing $k(n)^*(\mathrm{Gr}_d(\mathbb{R}^m))$ collapses at E_2;
- $k_n(\mathrm{Gr}_d(\mathbb{R}^{2^{n+1}-\epsilon+2l}))$ is polynomial of degree $\lfloor d/2 \rfloor$ as a function of l.

Known rational calculations imply that the conjecture is true when $n = 0$. It is also easy to show that the conjecture is true when $d = 1$, and one calculates

$$k_n(\mathrm{Gr}_1(\mathbb{R}^m)) = \begin{cases} m & \text{if } 1 \leq m \leq 2^{n+1}, \\ 2^{n+1} - \epsilon & \text{if } m = 2^{n+1} - \epsilon + 2l. \end{cases}$$

With much more work we prove the following.

Theorem 1.8 *Conjecture 1.7 is true when $d = 2$. Thus the Atiyah–Hirzebruch spectral sequence computing $K(n)^*(\mathrm{Gr}_2(\mathbb{R}^m))$ collapses at $E_{2^{n+1}}$, the Adams spectral sequence computing $k(n)^*(\mathrm{Gr}_2(\mathbb{R}^m))$ collapses at E_2, and we have the calculation*

$$k_n(\mathrm{Gr}_2(\mathbb{R}^m)) = \begin{cases} \binom{m}{2} & \text{if } 2 \leq m \leq 2^{n+1}, \\ \binom{2^{n+1}-\epsilon}{2} + l & \text{if } m = 2^{n+1} - \epsilon + 2l. \end{cases}$$

We are firm believers in our conjectures. For more evidence, the second author has made extensive computer calculations verifying Conjecture 1.7 in hundreds more cases with larger values of d; see the tables in the appendix.

For $d \geq 2$, computing the size of $H^*(\mathrm{Gr}_d(\mathbb{R}^m); Q_n)$ seems tricky. We have organized our efforts by studying how these numbers change as m is increased as follows.

Let $C_d(\mathbb{R}^m)$ denote the cofiber of the inclusion $\mathrm{Gr}_d(\mathbb{R}^{m-1}) \to \mathrm{Gr}_d(\mathbb{R}^m)$, so there is a cofiber sequence $\mathrm{Gr}_d(\mathbb{R}^{m-1}) \xrightarrow{i} \mathrm{Gr}_d(\mathbb{R}^m) \xrightarrow{p} C_d(\mathbb{R}^m)$. In Section 3.3, $C_d(\mathbb{R}^m)$ is identified as the Thom space of the canonical normal bundle over $\mathrm{Gr}_{d-1}(\mathbb{R}^{m-1})$, and in Section 3.4 we study the Q_n–module

$\widetilde{H}^*(C_d(\mathbb{R}^m);\mathbb{Z}/2)$, viewed as $H^*(\mathrm{Gr}_{d-1}(\mathbb{R}^{m-1});\mathbb{Z}/2)$ equipped with an explicit twisted Q_n–action. One has an induced short exact sequence of modules over the Steenrod algebra

$$0\to \widetilde{H}^*(C_d(\mathbb{R}^m);\mathbb{Z}/2)\xrightarrow{p^*} H^*(\mathrm{Gr}_d(\mathbb{R}^m);\mathbb{Z}/2)\xrightarrow{i^*} H^*(\mathrm{Gr}_d(\mathbb{R}^{m-1});\mathbb{Z}/2)\to 0,$$

inducing a long exact sequence on Q_n–homology.

When m is even, we see much orderly behavior.

Theorem 1.9 *Let m be even.*

(a) $H^{d(m-d)}(\mathrm{Gr}_d(\mathbb{R}^m);Q_n)\simeq \mathbb{Z}/2$, *ie the nonzero top-dimensional cohomology class is not in the image of Q_n for all n.*

(b) *The chain complex $\big(\widetilde{H}^*(C_d(\mathbb{R}^m);Q_n)\big)$ is dual to the chain complex*

$$\big(H^{d(m-d)-*}(\mathrm{Gr}_{d-1}(\mathbb{R}^{m-1});Q_n)\big).$$

(c) *If Conjecture 1.7 is true for $(n,d,m-1)$ and $(n,d-1,m-1)$, then it is true for (n,d,m). Furthermore, $\mathrm{Gr}_d(\mathbb{R}^m)$ will then be $k(n)$–oriented, and the cofiber sequence above will induce short exact sequences*

$$0\to \widetilde{H}^*(C_d(\mathbb{R}^m);Q_n)\xrightarrow{p^*} H^*(\mathrm{Gr}_d(\mathbb{R}^m);Q_n)\xrightarrow{i^*} H^*(\mathrm{Gr}_d(\mathbb{R}^{m-1});Q_n)\to 0,$$
$$0\to \widetilde{K}(n)^*(C_d(\mathbb{R}^m))\xrightarrow{p^*} K(n)^*(\mathrm{Gr}_d(\mathbb{R}^m))\xrightarrow{i^*} K(n)^*(\mathrm{Gr}_d(\mathbb{R}^{m-1}))\to 0.$$

We prove Theorem 1.9 in Section 4. We make use of the additive basis $\{s_\lambda\}$ dual to the classical Schubert cells. Here λ runs through partitions having at most d parts, each no bigger than $m-d$. In [9], Cristian Lennart gave a combinatorial formula for $Q_n(s_\lambda)$, and we use this to prove (a). Duality statement (b) follows quite formally from (a), and (c) follows easily from (b).

When m is odd, the analogues of statements (a) and (b) are false, and, for $d\geq 3$, the full behavior of the connecting map in the Q_n–homology long exact sequence,

$$\delta\colon H^*(\mathrm{Gr}_d(\mathbb{R}^{m-1});Q_n)\to \widetilde{H}^{*+2^{n+1}-1}(C_d(\mathbb{R}^m);Q_n),$$

is as yet unclear to the authors. In Section 6, we will prove analogues of Theorems 1.1 and 1.2 for $C_d(\mathbb{R}^m)$, and then speculate on behavior of δ that would be compatible with all of our computations.

However, when $d=2$, we have the following result.

Theorem 1.10 *Let $m>2^{n+1}$ be odd. Then $k_{Q_n}(C_2(\mathbb{R}^m))=2^{n+1}-2$ and the map*

$$\widetilde{H}^*(C_2(\mathbb{R}^m);Q_n)\xrightarrow{p^*} H^*(\mathrm{Gr}_2(\mathbb{R}^m);Q_n)$$

is zero, so there is a short exact sequence

$$0\to H^*(\mathrm{Gr}_2(\mathbb{R}^m);Q_n)\xrightarrow{i^*} H^*(\mathrm{Gr}_2(\mathbb{R}^{m-1});Q_n)\xrightarrow{\delta} \widetilde{H}^{*+2^{n+1}-1}(C_2(\mathbb{R}^m);Q_n)\to 0.$$

From this, Theorem 1.8 quickly follows and one can deduce that, in this case, there is a short exact sequence

$$0 \to K(n)^*(\mathrm{Gr}_2(\mathbb{R}^m)) \xrightarrow{i^*} K(n)^*(\mathrm{Gr}_2(\mathbb{R}^{m-1})) \xrightarrow{\delta} \widetilde{K}(n)^{*+1}(C_2(\mathbb{R}^m)) \to 0.$$

We prove Theorem 1.10 in Section 5. The tools we use are very different from those used in proving Theorem 1.9: we work with the classical presentation of $H^*(\mathrm{Gr}_d(\mathbb{R}^m); \mathbb{Z}/2)$ as a ring of Stiefel–Whitney classes.

1.3 Comparison with other work

When comparing our work to what has come before, the first thing to say is that the outcome of our calculations — though not the methods — are in line with the classical calculations first made by C Ehresmann in 1937 [3]. He determined the additive structure of both $H^*(\mathrm{Gr}_d(\mathbb{R}^m); \mathbb{Z}/2)$ and $H^*(\mathrm{Gr}_d(\mathbb{R}^m); \mathbb{Q})$. He also showed that all the torsion in $H^*(\mathrm{Gr}_d(\mathbb{R}^m); \mathbb{Z})$ was of order 2; in modern terms this is equivalent to showing that the Bockstein spectral sequence computing $H^*(\mathrm{Gr}_d(\mathbb{R}^m); \mathbb{Z})$ collapses after the first nonzero differential given by $Q_0 = \mathrm{Sq}^1 = \beta$.

Calculating the Morava K–theories of $\mathrm{Gr}_d(\mathbb{R}^\infty) = BO(d)$ was done first by Kono and Yagita [7], and then, with a simpler proof, by Kitchloo and Wilson [6]. Again, the AHSS computing $K(n)^*(BO(d))$ collapses after the first nonzero differential, but the collapsing is for an elementary reason: $H^*(BO(d); Q_n)$ is concentrated in even degrees. Indeed, one quickly learns that the complexification map $BO(d) \to BU(d)$ induces an epimorphism $K(n)^*(BU(d)) \to K(n)^*(BO(d))$, so $K(n)^*(BO(d))$ is generated by Chern classes $c_1, \dots, c_d$.

An equivalent statement is that $H^*(BO(d); Q_n)$ is generated by the classes $w_1^2, \dots, w_d^2$. These will still be permanent classes in the AHSS converging to $K(n)^*(\mathrm{Gr}_d(\mathbb{R}^m))$, but now we have odd-dimensional classes as well, with the number of these seemingly growing as d and m grow.

Finally, we point out that we do not attempt to describe $K(n)^*(\mathrm{Gr}_d(\mathbb{R}^m))$ as a $K(n)^*$–algebra. Our results do tell us something about this, however. In the situation of Theorem 1.1, the known algebra $H^*(\mathrm{Gr}_d(\mathbb{R}^m); \mathbb{Z}/2) \otimes K(n)^*$ will be an associated graded. Similarly, whenever our conjecture is valid, $H^*(\mathrm{Gr}_d(\mathbb{R}^m); Q_n) \otimes K(n)^*$ would be an associated graded of the $K(n)^*$–algebra $K(n)^*(\mathrm{Gr}_d(\mathbb{R}^m))$. What is still needed, and might be necessary to prove our conjectural collapsing in general, are sensible constructions of classes in odd degrees.

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2 The proofs of Theorems 1.1 and 1.2

In this section we prove Theorems 1.1 and 1.2 by using our chromatic fixed point theorem Theorem 1.4.

2.1 A fixed point formula

Let G be a finite group, and let V be an m–dimensional real representation of G. Then $\mathrm{Gr}_d(V)$, the space of d–planes in V, is a model for $\mathrm{Gr}_d(\mathbb{R}^m)$ with an evident G–action. Here we describe $\mathrm{Gr}_d(V)^G$, its space of G–fixed points.

To state this, we need some notation. Let $V_1, \dots, V_k$ be the irreducible real representations of G, let $r_i = \dim_{\mathbb{R}} V_i$, and let $\mathbb{D}_i = \mathrm{End}_{\mathbb{R}[G]}(V_i, V_i)$. Each of the endomorphism algebras $\mathbb{D}_i$ will be a finite-dimensional real division algebra, and thus isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, and $\dim_{\mathbb{R}} \mathbb{D}_i$ will divide r_i.

Proposition 2.1 *If $V = V_1^{m_1} \oplus \cdots \oplus V_k^{m_k}$, then there is a homeomorphism*

$$\mathrm{Gr}_d(V)^G = \coprod_{j_1 r_1 + \cdots + j_k r_k = d} \mathrm{Gr}_{j_1}(\mathbb{D}_1^{m_1}) \times \cdots \times \mathrm{Gr}_{j_k}(\mathbb{D}_k^{m_k}).$$

Proof The fixed point space $\mathrm{Gr}_d(V)^G$ will be the space of sub–G–modules $W < V$ of real dimension d. Such a G–module W will decompose canonically as $W = W_1 \oplus \cdots \oplus W_k$, with $W_i < V_i^{m_i}$. If $d_i = \dim_{\mathbb{R}} W_i$, then $d_1 + \cdots + d_k = d$. Thus we have a decomposition

$$\mathrm{Gr}_d(V)^G = \coprod_{d_1 + \cdots + d_k = d} \mathrm{Gr}_{d_1}(V_1^{m_1})^G \times \mathrm{Gr}_{d_2}(V_2^{m_2})^G \times \cdots \times \mathrm{Gr}_{d_k}(V_k^{m_k}).$$

A submodule W_i of $V_i^{m_i}$ must be isomorphic to V_i^j for some j; thus $\mathrm{Gr}_{d_i}(V_i^{m_1})^G$ will be empty unless $d_i = j_i r_i$ for some j_i.

Finally, using that $\mathrm{Hom}_{\mathbb{R}[G]}(V_i^{j_i}, V_i^{m_i}) = \mathrm{Hom}_{\mathbb{D}}(\mathbb{D}^{j_i}, \mathbb{D}^{m_i})$, one deduces that the submodules of $V_i^{m_i}$ isomorphic to $V_i^{j_i}$ correspond to the $\mathbb{D}$–subspaces of $\mathbb{D}^{m_i}$ of dimension j_i over $\mathbb{D}$. Thus there is a homeomorphism

$$\mathrm{Gr}_{j_i r_i}(V_i^{m_i})^G = \mathrm{Gr}_{j_i}(\mathbb{D}_i^{m_i}). \qquad \square$$

Corollary 2.2 *If $V = V_1^{m_1} \oplus \cdots \oplus V_k^{m_k}$, then, for any n,*

$$k_n(\mathrm{Gr}_d(V)^G) = \sum_{j_1 r_1 + \cdots + j_k r_k = d} k_n(\mathrm{Gr}_{j_1}(\mathbb{D}_1^{m_1})) \cdots k_n(\mathrm{Gr}_{j_k}(\mathbb{D}_k^{m_k})).$$

Proof A consequence of the Künneth theorem for $K(n)_*$ is that $k_n(X \times Y) = k_n(X) k_n(Y)$. Thus the corollary follows from the proposition. $\square$

Remark 2.3 If $\mathbb{D} = \mathbb{C}$ or $\mathbb{H}$, then $\mathrm{Gr}_d(\mathbb{D}^m)$ has a CW structure with $\binom{m}{d}$ cells that are all even-dimensional, and thus $k_n(\mathrm{Gr}_d(\mathbb{D}^m)) = \binom{m}{d}$ for all n.

2.2 Proof of Theorem 1.1

Theorem 1.1 says that if $m \le 2^{n+1}$ then $k_n(\mathrm{Gr}_d(\mathbb{R}^m)) = \binom{m}{d}$. Using Theorem 1.4 and Proposition 2.1, we prove this by induction on n.

The $n = 0$ case of the theorem is easy to check, as

$$\mathrm{Gr}_d(\mathbb{R}^0) = \begin{cases} * & \text{if } d = 0, \\ \varnothing & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \mathrm{Gr}_d(\mathbb{R}^1) = \begin{cases} * & \text{if } d = 0, 1, \\ \varnothing & \text{otherwise.} \end{cases}$$

For the inductive step, assume that if $p \le 2^n$ then $k_{n-1}(\mathrm{Gr}_d(\mathbb{R}^p)) = \binom{p}{d}$.

Let $m \le 2^{n+1}$. As it is clear that $k_n(\mathrm{Gr}_d(\mathbb{R}^m)) \le \binom{m}{d}$, our goal is to show that $k_n(\mathrm{Gr}_d(\mathbb{R}^m)) \ge \binom{m}{d}$.

Let C_2 be the cyclic group of order 2. To get our needed lower bound, our strategy will be to make $\mathbb{R}^m$ into a C_2–module, and then apply Theorem 1.4.

The group C_2 has two irreducible 1–dimensional real representations; call them L_1 and L_2. Since $m \le 2^{n+1}$, we can write m as $m = p + q$ with both $p \le 2^n$ and $q \le 2^n$. Now let $V = L_1^p \oplus L_2^q$, an m–dimensional real representation of C_2.

Applying Proposition 2.1, we see that

$$\mathrm{Gr}_d(V)^{C_2} = \coprod_{i+j=d} \mathrm{Gr}_i(\mathbb{R}^p) \times \mathrm{Gr}_j(\mathbb{R}^q).$$

Applying Theorem 1.4 to this, we learn that

$$\begin{aligned} k_n(\mathrm{Gr}_d(\mathbb{R}^m)) &\ge \sum_{i+j=d} k_{n-1}(\mathrm{Gr}_i(\mathbb{R}^p)) k_{n-1}(\mathrm{Gr}_j(\mathbb{R}^q)) \\ &= \sum_{i+j=d} \binom{p}{i}\binom{q}{j} \quad \text{(by inductive hypothesis)} \\ &= \binom{m}{d}. \end{aligned}$$

Remark 2.4 The same inductive proof can be used to prove the classical result that

$$\dim_{\mathbb{Z}/2} H^*(\mathrm{Gr}_d(\mathbb{R}^m); \mathbb{Z}/2) = \binom{m}{d}$$

for all m and d, with our chromatic fixed point theorem Theorem 1.4 replaced by the classical theorem of Ed Floyd [4, Theorem 4.4]: if the cyclic group C_p acts on a finite CW complex X, then $\dim_{\mathbb{Z}/p} H^*(X; \mathbb{Z}/p) \ge \dim_{\mathbb{Z}/p} H^*(X^{C_p}; \mathbb{Z}/p)$. It would be interesting to know if this argument was known to Floyd, or others, like Bob Stong, who regularly worked with these sorts of group actions.

2.3 Proof of Theorem 1.2

The strategy of the proof of Theorem 1.2 is the same as the proof in the last subsection: we get a lower bound on $k_n(\mathrm{Gr}_d(\mathbb{R}^m))$ by letting a cyclic 2–group act on $\mathbb{R}^m$ and applying Theorem 1.4.

In this case, the representation theory of C_2 is not rich enough to give us a big enough lower bound, but a well chosen real representation of the group C_4 of order 4 works better. Curiously, in our calculation of k_{n-1} of the resulting fixed point space, we are able to use our already proven Theorem 1.1, so the proof is not by induction, but more direct.

The group C_4 has three irreducible real representations: L_1 and L_2 of dimension 1, and R of real dimension 2. Note that $\mathrm{End}_{\mathbb{R}[C_4]}(R) \simeq \mathbb{C}$.

Now let $m = 2^{n+1} - \epsilon + 2l$ with $\epsilon = 0$ or 1, and $l \geq 0$. We define an m–dimensional real representation V of C_4 by $V = L_1^{2^n} \oplus L_2^{2^n - \epsilon} \oplus R^l$.

Applying Proposition 2.1, we see that

$$\mathrm{Gr}_d(V)^{C_4} = \coprod_{j+k+2i=d} \mathrm{Gr}_j(\mathbb{R}^{2^n}) \times \mathrm{Gr}_k(\mathbb{R}^{2^n-\epsilon}) \times \mathrm{Gr}_i(\mathbb{C}^l).$$

Applying Theorem 1.4 to this, we learn that

$$\begin{aligned} k_n(\mathrm{Gr}_d(\mathbb{R}^m)) &\geq \sum_{j+k+2i=d} k_{n-1}(\mathrm{Gr}_j(\mathbb{R}^{2^n}))k_{n-1}(\mathrm{Gr}_k(\mathbb{R}^{2^n-\epsilon}))k_{n-1}(\mathrm{Gr}_i(\mathbb{C}^l)) \\ &= \sum_{j+k+2i=d} \binom{2^n}{j}\binom{2^n-\epsilon}{k}\binom{l}{i} \quad \text{(using Theorem 1.1)} \\ &= \sum_i \Bigl[\sum_{j+k=d-2i} \binom{2^n}{j}\binom{2^n-\epsilon}{k}\Bigr]\binom{l}{i} \\ &= \sum_i \binom{2^{n+1}-\epsilon}{d-2i}\binom{l}{i}. \end{aligned}$$

3 The Q_n homology of $\mathrm{Gr}_d(\mathbb{R}^m)$: background material

3.1 The AHSS and the ASS for Morava K–theory

Let $n \geq 1$. We recall the structure of the AHSS converging to $K(n)^*(X)$ (as always, in this paper, with $p = 2$). It is a spectral sequence of graded $K(n)^* = \mathbb{Z}/2[v_n^{\pm}]$ algebras with

$$E_2^{*,\star}(X) = H^*(X; K(n)^\star) = H^*(X; \mathbb{Z}/2)[v_n^{\pm}].$$

Here v_n has cohomological degree $2 - 2^{n+1}$.

Sparseness of the rows implies that the differential d_r will be zero unless $r = s(2^{n+1}-2)+1$ for some s. The first possible nonzero differential, $d_{2^{n+1}-1}$, satisfies the following formula [15]: given $x \in E_2^{*,0}(X) = H^*(X;\mathbb{Z}/2)$,

$$d_{2^{n+1}-1}(x) = Q_n(x)v_n.$$

It follows that $E_{2^{n+1}}(X) \simeq H^*(X;Q_n)[v_n^{\pm}]$, and so the dimension of $E_{2^{n+1}}(X)$ as a $K(n)^*$–vector space will equal $k_{Q_n}(X)$, the dimension of the Q_n–homology of X. One immediately deduces part of Lemma 1.5: $k_{Q_n}(X) = k_n(X)$ if and only if the AHSS converging to $K(n)^*(X)$ collapses at $E_{2^{n+1}}(X)$.

To continue with the proof of Lemma 1.5, let $cE_r^{*,\star}(X)$ denote the terms of the AHSS computing $k(n)^*(X)$, a 4th quadrant spectral sequence. Note that $cE_2^{*,\star} = H^*(X;\mathbb{Z}/2)[v_n]$ embeds in $E_2^{*,\star}(X) = H^*(X;\mathbb{Z}/2)[v_n^{\pm}]$, and equals it for $\star \leq 0$, and that the latter spectral sequence is obtained from the former by inverting v_n.

It follows that $cE_{2^{n+1}}^{*,\star}(X) = E_{2^{n+1}}^{*,\star}(X)$ for $\star < 0$, with the map on the 0–line between the spectral sequences corresponding to the epimorphism $Z^*(X;Q_n) \twoheadrightarrow H^*(X;Q_n)$. From this, one sees that any higher differential in the $k(n)^*(X)$ AHSS would be detected in the $K(n)^*(X)$ AHSS. Since this second spectral sequence is the localization of the first, we can conclude that the $K(n)^*(X)$ AHSS collapses at $E_{2^{n+1}}(X)$ if and only if the $k(n)^*(X)$ AHSS collapses at $cE_{2^{n+1}}(X)$.

Next we note that the AHSS spectral sequence $cE_r^{*,\star}(X)$ identifies with the ASS computing $k(n)^*(X)$ with suitable reindexing, with $cE_{2^{n+1}}^{*,\star}(X)$ corresponding to the Adams E_2 term. Firstly, a result of C R F Maunder [11] implies that the AHSS converging to $[X,k(n)]_*$ can be constructed by taking the Postnikov filtration of the spectrum $k(n)$. But the Postnikov tower for $k(n)$ is also an Adams tower: as described in the survey paper [14, Section 5], there is a cofibration sequence

$$\Sigma^{2^{n+1}-2}k(n) \xrightarrow{v_n} k(n) \xrightarrow{\pi} H\mathbb{Z}/2 \xrightarrow{\bar{Q}_n} \Sigma^{2^{n+1}-1}k(n)$$

such that $\Sigma^{2^{n+1}-1}\pi \circ \bar{Q}_n = Q_n$ and π induces the epimorphism $A \to A/AQ_n$ on mod 2 cohomology.

Finally, we note that, when $n=0$, one still has the cofibration sequence as above, now with $v_0 = 2$, so that the ASS for $k(0) = H\mathbb{Z}$ is similarly related to the Bockstein spectral sequence.

3.2 The description of $H^*(\mathrm{Gr}_d(\mathbb{R}^m);\mathbb{Z}/2)$ via Stiefel–Whitney classes

We recall classical results that are either explicitly in [12] or can easily be deduced from the material there.

Let $w_1,\dots,w_d$ denote the Stiefel–Whitney classes of the canonical d–dimensional bundle γ_d over $\mathrm{Gr}_d(\mathbb{R}^\infty)$. One has

$$H^*(\mathrm{Gr}_d(\mathbb{R}^\infty);\mathbb{Z}/2) = \mathbb{Z}/2[w_1,\dots,w_d].$$

Dual classes $\bar{w}_1, \bar{w}_2, \dots$ are defined by the equation

$$(1 + w_1 + \cdots + w_d)(1 + \bar{w}_1 + \bar{w}_2 + \cdots) = 1,$$

and this allows one to write the classes $\bar{w}_k$ as polynomials in $w_1, \dots, w_d$.

The inclusion $\mathrm{Gr}_d(\mathbb{R}^m) \hookrightarrow \mathrm{Gr}_d(\mathbb{R}^\infty)$ then induces a surjective ring homomorphism

$$H^*(\mathrm{Gr}_d(\mathbb{R}^\infty); \mathbb{Z}/2) \to H^*(\mathrm{Gr}_d(\mathbb{R}^m); \mathbb{Z}/2)$$

with kernel $J(d, m-d) = (\bar{w}_k \mid k > m-d)$. Now $\bar{w}_k$ can be interpreted as $w_k(\gamma_d^\perp)$, where $\gamma_d^\perp$ is the $(m-d)$–dimensional bundle complementary to γ_d.

We record some useful consequences. To state these, it is useful to let

$$i : \mathrm{Gr}_d(\mathbb{R}^{m-1}) \hookrightarrow \mathrm{Gr}_d(\mathbb{R}^m)$$

be the inclusion induced by the inclusion $\mathbb{R}^{m-1} \hookrightarrow \mathbb{R}^m$, and to let

$$j : \mathrm{Gr}_{d-1}(\mathbb{R}^{m-1}) \hookrightarrow \mathrm{Gr}_d(\mathbb{R}^m)$$

be the inclusion sending $V \subset \mathbb{R}^{m-1}$ to $V \oplus \mathbb{R} \subset \mathbb{R}^m$.

Lemma 3.1 (a) *The ideal $J(d, m-d)$ is generated by the d classes $\bar{w}_{m-d+1}, \bar{w}_{m-d+2}, \dots, \bar{w}_m$.*

(b) *In $H^*(\mathrm{Gr}_d(\mathbb{R}^m); \mathbb{Z}/2)$, $w_d \bar{w}_{m-d} = 0$.*

(c) $\ker\{i^*\} = (\bar{w}_{m-d}) \subset H^*(\mathrm{Gr}_d(\mathbb{R}^m); \mathbb{Z}/2)$.

(d) $\ker\{j^*\} = (w_d) \subset H^*(\mathrm{Gr}_d(\mathbb{R}^m); \mathbb{Z}/2)$.

Proof Statement (a) follows from the recursive relations among the $\bar{w}_k$. Statement (b) follows from the equation

$$(1 + w_1 + \cdots + w_d)(1 + \bar{w}_1 + \cdots + \bar{w}_{m-d}) = 1,$$

which holds in $H^*(\mathrm{Gr}_d(\mathbb{R}^m); \mathbb{Z}/2)$. Statement (c) follows from the fact that

$$J(d, m-1-d) = J(d, m) + (\bar{w}_{m-d}),$$

and (d) follows from (c), noting that j can be written as the composite

$$\mathrm{Gr}_{d-1}(\mathbb{R}^{m-1}) \simeq \mathrm{Gr}_{m-d}(\mathbb{R}^{m-1}) \xrightarrow{i} \mathrm{Gr}_{m-d}(\mathbb{R}^m) \simeq \mathrm{Gr}_d(\mathbb{R}^m),$$

where the indicated homeomorphisms are given by taking complementary subspaces (and, in cohomology, these maps swap each w_i with a $\bar{w}_j$). □

We end this subsection with a couple more facts about $H^*(\mathrm{Gr}_d(\mathbb{R}^m); \mathbb{Z}/2)$.

An additive basis for $H^q(\mathrm{Gr}_d(\mathbb{R}^m);\mathbb{Z}/2)$ is given by the monomials

$$\left\{ w_1^{r_1} w_2^{r_2} \cdots w_d^{r_d} \;\middle|\; \sum_{i=1}^{d} r_i \le m-d \right\},$$

so the top-dimensional class is w_d^{m-d} in degree $d(m-d)$; see [5].

The Wu formulae [12, page 94] are closed formulae for $\mathrm{Sq}^i w_j$, and, in theory, formulae for $Q_n(w_j)$ follow.

3.3 A description of the cofiber $C_d(\mathbb{R}^m)$ and its cohomology

Recall that $C_d(\mathbb{R}^m)$ is defined as the cofiber of the inclusion $\mathrm{Gr}_d(\mathbb{R}^{m-1}) \xrightarrow{i} \mathrm{Gr}_d(\mathbb{R}^m)$. This cofiber can be identified as a Thom space as follows.

Proposition 3.2 *Let $S(\gamma_{d-1}^{\perp})$ and $D(\gamma_{d-1}^{\perp})$ be the sphere and disk bundles associated to*

$$\gamma_{d-1}^{\perp} \to \mathrm{Gr}_{d-1}(\mathbb{R}^{m-1}).$$

There is a pushout

$$\begin{tikzcd}
S(\gamma_{d-1}^{\perp}) \arrow[r, "f"] \arrow[d] & \mathrm{Gr}_d(\mathbb{R}^{m-1}) \arrow[d] \\
D(\gamma_{d-1}^{\perp}) \arrow[r, "f"] & \mathrm{Gr}_d(\mathbb{R}^m),
\end{tikzcd}$$

inducing a homeomorphism $f\colon \mathrm{Th}(\gamma_{d-1}^{\perp}) \xrightarrow{\sim} C_d(\mathbb{R}^m)$, such that the composite

$$\mathrm{Gr}_{d-1}(\mathbb{R}^{m-1}) \xrightarrow[\sim]{\text{0–section}} D(\gamma_{d-1}^{\perp}) \xrightarrow{f} \mathrm{Gr}_d(\mathbb{R}^m)$$

is the map j of Lemma 3.1.

Proof Recall that

$$D(\gamma_{d-1}^{\perp}) = \{(V,v) \mid V \in \mathrm{Gr}_{d-1}(\mathbb{R}^{m-1}), v \in V^{\perp}, |v| \le 1\},$$
$$S(\gamma_{d-1}^{\perp}) = \{(V,v) \mid V \in \mathrm{Gr}_{d-1}(\mathbb{R}^{m-1}), v \in V^{\perp}, |v| = 1\}.$$

We define $f\colon D(\gamma_{d-1}^{\perp}) \to \mathrm{Gr}_d(\mathbb{R}^m)$ by the formula

$$f(V,v) = V + \langle v + \sqrt{1-|v|^2}\,e_m \rangle,$$

where e_m is the m^{th} standard basis vector in $\mathbb{R}^m$. We claim this f has the needed properties.

First, note that $f(V,\mathbb{0}) = V + \langle e_m \rangle = V \oplus \mathbb{R} = j(V)$.

Second, $(V,v) \in S(\gamma_{d-1}^{\perp})$ if and only if $f(V,v) = V + \langle v \rangle$, and so is an element of $\mathrm{Gr}_d(\mathbb{R}^{m-1})$. Furthermore, $f\colon S(\gamma_{d-1}^{\perp}) \to \mathrm{Gr}_d(\mathbb{R}^{m-1})$ is surjective: given any $W \in \mathrm{Gr}_d(\mathbb{R}^{m-1})$, if we choose any $(d-1)$–dimensional subspace V of W, and a unit length vector $v \in W$ in the 1–dimensional orthogonal complement, then $f(V,v) = W$.

Finally, we need to check that f is bijective on $\overset{\circ}{D}(\gamma_{d-1}^{\perp}) = D(\gamma_{d-1}^{\perp}) - S(\gamma_{d-1}^{\perp})$. To check this, let $W \in \mathrm{Gr}_d(\mathbb{R}^m)$ be a d–dimensional subspace of $\mathbb{R}^m$ not contained in $\mathbb{R}^{m-1}$, so that

$$V = W \cap \mathbb{R}^{m-1} \in \mathrm{Gr}_{d-1}(\mathbb{R}^{m-1}).$$

Let $V^{\perp}$ be the complement of V in of $\mathbb{R}^m$ so that $W \cap V^{\perp}$ is one-dimensional, and let v be the unique unit vector $v \in W \cap V^{\perp}$ such that v has positive m^{th} coordinate. Let $\pi\colon \mathbb{R}^m \to \mathbb{R}^{m-1}$ be the standard projection. We claim that $f(V, \pi(v)) = W$ and $(V, \pi(v))$ is the unique point in $\overset{\circ}{D}(\gamma_{d-1}^{\perp})$ with this property: since $|v| = 1$ the m^{th} component of v is $\sqrt{1 - |\pi(v)|^2}$; thus $v = \pi(v) + \sqrt{1 - |\pi(v)|^2} e_m$ and so $f(V, \pi(v)) = V + \langle v \rangle = W$. □

Let $u_{\gamma_{d-1}^{\perp}} \in \widetilde{H}^{m-d}(\mathrm{Th}(\gamma_{d-1}^{\perp}))$ be the Thom class of $\gamma_{d-1}^{\perp} \to \mathrm{Gr}_{d-1}(\mathbb{R}^{m-1})$. Then $\widetilde{H}^{m-d}(\mathrm{Th}(\gamma_{*}^{\perp}))$ is a free rank 1 $H^*(\mathrm{Gr}_{d-1}(\mathbb{R}^{m-1}))$–module on $u_{\gamma_{d-1}^{\perp}}$. Meanwhile, $H^*(\mathrm{Gr}_d(\mathbb{R}^m))$ is a $H^*(\mathrm{Gr}_{d-1}(\mathbb{R}^{m-1}))$–module via j^*, and the ideal $(\bar{w}_{m-d}) = \widetilde{H}^*(C_d(\mathbb{R}^m))$ is a submodule. The proposition thus implies the following.

Corollary 3.3 *The map* $f^*\colon \widetilde{H}^*(C_d(\mathbb{R}^m)) \xrightarrow{\sim} \widetilde{H}^*(\mathrm{Th}(\gamma_{d-1}^{\perp}))$ *is an isomorphism of free rank* 1 $H^*(\mathrm{Gr}_{d-1}(\mathbb{R}^{m-1}))$*–modules, and* $f^*(\bar{w}_{m-d}) = u_{\gamma_{d-1}^{\perp}}$.

3.4 The characteristic class associated to Q_n and a twisted Q_n–module

Let $\alpha_n \in H^{2^{n+1}-1}(\mathrm{Gr}_{d-1}(\mathbb{R}^{m-1}))$ be defined as the element satisfying

$$Q_n(\bar{w}_{m-d}) = \alpha_n \bar{w}_{m-d} \in \widetilde{H}^*(C_d(\mathbb{R}^m)).$$

Then define

$$\hat{Q}_n\colon H^*(\mathrm{Gr}_{d-1}(\mathbb{R}^{m-1})) \to H^{*+2^{n+1}-1}(\mathrm{Gr}_{d-1}(\mathbb{R}^{m-1}))$$

by the formula

$$\hat{Q}_n(x) = Q_n(x) + x\alpha_n.$$

Proposition 3.4 $\hat{Q}_n^2 = 0$, *and the chain complex* $\big(H^*(\mathrm{Gr}_{d-1}(\mathbb{R}^{m-1})), \hat{Q}_n\big)$ *is isomorphic to the chain complex* $\big(\widetilde{H}^{*+m-d}(C_d(\mathbb{R}^m)), Q_n\big)$.

Proof Let $\Theta\colon H^*(\mathrm{Gr}_{d-1}(\mathbb{R}^{m-1})) \to \widetilde{H}^{*+m-d}(C_d(\mathbb{R}^m))$ be the isomorphism established in the last subsection, $\Theta(x) = x\bar{w}_{m-d}$. The proposition follows once we check that $\Theta(\hat{Q}_n(x)) = Q_n(\Theta(x))$;

$$\begin{aligned}\Theta(\hat{Q}_n(x)) &= \hat{Q}_n(x)\bar{w}_{m-d} \\ &= (Q_n(x) + x\alpha_n)\bar{w}_{m-d} \\ &= Q_n(x)\bar{w}_{m-d} + x(\alpha_n\bar{w}_{m-d}) \\ &= Q_n(x)\bar{w}_{m-d} + xQ_n(\bar{w}_{m-d}) \\ &= Q_n(x\bar{w}_{m-d}) \\ &= Q_n(\Theta(x)).\end{aligned}$$

□

It is useful to put the class α_n in context. Given any element a in the Steenrod algebra $\mathcal{A}$, one gets a characteristic class $w_a(\xi) \in H^{|a|}(B;\mathbb{Z}/2)$ associated to any real vector bundle $\xi \to B$; $w_a(\xi)$ is defined as the element satisfying $a(u_\xi) = w_a(\xi)u_\xi \in \widetilde{H}^{\dim\xi+|a|}(\mathrm{Th}(\xi);\mathbb{Z}/2)$, where u_ξ is the Thom class of ξ. So, for example, $w_{\mathrm{Sq}^n}(\xi) = w_n(\xi)$, and, relevant for us, our class α_n equals $w_{Q_n}(\xi)$ when $\xi = \gamma_{d-1}^{\perp} \to \mathrm{Gr}_{d-1}(\mathbb{R}^{m-1})$.

We have the following characterization of w_{Q_n}.

Proposition 3.5 *w_{Q_n} is the unique characteristic class satisfying the following two properties*:

(a) $w_{Q_n}(\xi \oplus \nu) = w_{Q_n}(\xi) + w_{Q_n}(\nu)$;

(b) *if $\gamma \to B$ is one-dimensional, then* $w_{Q_n}(\gamma) = w_1(\gamma)^{2^{n+1}-1}$.

Proof Property (a) follows from the fact that Q_n is primitive in $\mathcal{A}$ (or, equivalently, that Q_n acts a derivation). To see property (b), one first calculates that $Q_n(t) = t^{2^n+1} \in \mathbb{Z}/2[t] = H^*(\mathbb{R}P^\infty;\mathbb{Z}/2)$, recalling that $Q_0 = \mathrm{Sq}^1$, and $Q_n = \mathrm{Sq}^{2^n}Q_{n-1} + Q_{n-1}\mathrm{Sq}^{2^n}$. Then property (b) follows, since if γ is the universal line bundle over $\mathbb{R}P^\infty$, then $u_\gamma = t$. Uniqueness follows from the splitting principle. □

Remark 3.6 Thus $w_{Q_n}(\xi)$ agrees with the "s–class" $s_{2^{n+1}-1}(\xi)$, analogous to the class of the same name for complex vector bundles as defined in [12, Section 16]. (These s_I are *not* the same as the s_λ of the next subsection; these are two conflicting and standard usages.)

Since $\gamma_{d-1}^{\perp} \oplus \gamma_{d-1}$ is trivial, property (b) has the following consequence.

Corollary 3.7 $$\alpha_n = w_{Q_n}(\gamma_{d-1}) \in H^{2^{n+1}-1}(\mathrm{Gr}_{d-1}(\mathbb{R}^{m-1});\mathbb{Z}/2).$$

3.5 The description of $H^*(\mathrm{Gr}_d(\mathbb{R}^m);\mathbb{Z}/2)$ via Schubert cells, and Lenart's formula

For the purposes of proving Theorem 1.9, we use an alternative description of $H^*(\mathrm{Gr}_d(\mathbb{R}^m);\mathbb{Z}/2)$.

We recall the cell structure of $\mathrm{Gr}_d(\mathbb{R}^{d+c})$ as described in [12, Section 6]. A *Schubert symbol* $\lambda = (\lambda_1,\dots,\lambda_d)$ of $\mathrm{Gr}_d(\mathbb{R}^m)$ is a sequence of integers

$$m-d \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d > 0.$$

The *weight* of λ is defined to be $\sum_i \lambda_i$ and is denoted $|\lambda|$. Such a λ is a partition contained inside of a $d \times (m-d)$ grid when depicted as Young diagrams — diagrams with λ_i boxes in the i^{th} row.

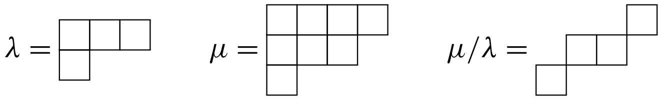
To each such partition is associated a Schubert cell $e(\lambda)$ of dimension λ in $\mathrm{Gr}_d(\mathbb{R}^m)$ defined by

$$e(\lambda) = \{V \in \mathrm{Gr}_d(\mathbb{R}^m) \mid \dim(V \cap \mathbb{R}^{i+\lambda_{d+1-i}}) \geq i \text{ for } 1 \leq i \leq d\}.$$

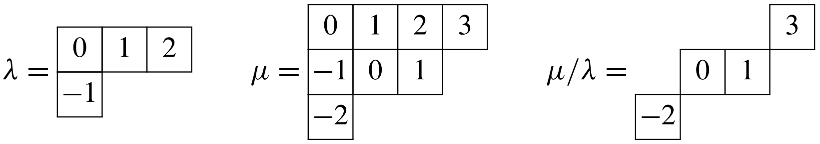
This cell decomposition of the Grassmannian leads to the dual Schubert cell basis for $H^*(\mathrm{Gr}_d(\mathbb{R}^m);\mathbb{Z}/2)$ with basis elements $s_\lambda \in H^{|\lambda|}(\mathrm{Gr}_d(\mathbb{R}^m);\mathbb{Z}/2)$.

With this notation, one has that $w_i = s_{(1^i)}$ and $\bar{w}_j = s_{(j)}$. Although we don't use this here, it is worth noting that the cohomology ring structure in this basis is described by the Littlewood–Richardson rule of symmetric function theory.

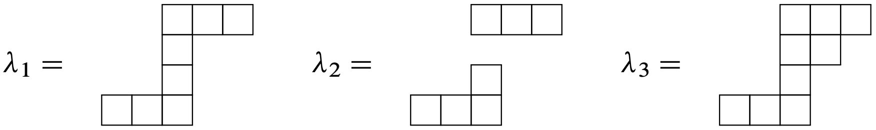
To state Lenart's formula for calculating Q_n on a Schubert basis element [9], we need some combinatorial definitions. Given a Young diagram λ that includes into another Young diagram μ, one can form the complement μ/λ. For example,



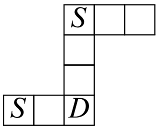
The *content* of a box b of μ in row i and column j is defined to be $\mathrm{c}(b) = j - i$. For a box b in the skew shape μ/λ, we define its content to be the content of b embedded in μ. Here we fill in the contents of the diagrams from above:



A skew-shape is said to be *connected* when each pair of boxes in the diagram is connected by a sequence of boxes that each share an edge. A shape λ is called a *border strip*, if it is connected and does not contain a 2×2 block of boxes. A shape satisfying just the second condition is called a *broken border* strip, and in particular, a border strip is an example of a broken border strip with just one connected component. If λ is a broken border strip, then we denote by $\mathrm{cc}(\lambda)$ the number of connected components of λ. If λ is not a broken border strip, then we define $\mathrm{cc}(\lambda) = \infty$. For example, in the next diagram, λ_1 is a border strip, λ_2 is a broken border strip that is not a border strip, and λ_3 is an example of a shape that is neither:



A *sharp corner* of a broken border strip is a box with no north, no west and no northwest neighbors. A *dull corner* is a box with both north and west neighbors, but no northwest neighbor. Let $\mathrm{C}(\mu/\lambda)$ denote the set of sharp and dull corners of μ/λ. For example, in the following diagram the sharp corners have been labeled S and the dull corners have been labeled D:



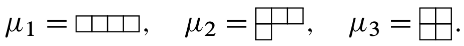
We are now ready to state Lenart's formula from [9]:

$$Q_n(s_\lambda) = \sum_{\substack{\mu \supset \lambda : |\mu| - |\lambda| = 2^{n+1}-1 \\ cc(\mu/\lambda) \leq 2}} d_{\lambda\mu} s_\mu, \tag{3-1}$$

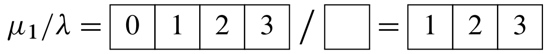
where μ/λ must be a broken border strip and

$$d_{\lambda\mu} = \begin{cases} \sum_{b \in C(\mu/\lambda)} c(b) & \text{if } \mu/\lambda \text{ is connected,} \\ 1 & \text{if } \mu/\lambda \text{ is disconnected.} \end{cases} \tag{3-2}$$

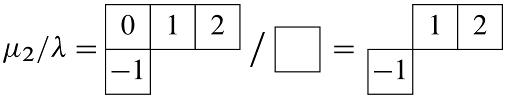
Example 3.8 As an example we compute Q_1 on $w_1 = s_\square$ in the Schubert basis in $\mathrm{Gr}_2(\mathbb{R}^6)$. There are three basis elements in degree four,



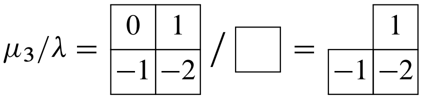
To compute $Q_n(s_\square)$ using (3-1) we must consider each complement. Let $\lambda = \square$. For μ_1,



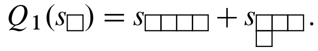
The complement is a border strip and there is just one sharp corner (the left most corner) and no dull corners. The content of the sharp corner is 1 modulo two; hence $d_{\lambda\mu_1} = 1$, and so s_{μ_1} is in the expansion of $Q_1(s_\lambda)$. Next we consider



This is a disconnected broken border strip; hence $d_{\lambda\mu_2} = 1$, and so s_{μ_2} is in the expansion. Finally,



There are two sharp corners, one of content -1 and the other of content 1. There is also one dull corner of content -2. This means $d_{\lambda\mu_3} = (-1) + 1 + 2 \equiv 0$, and so s_{μ_3} is not in the expansion. Hence,



4 Results about $H^*(\mathrm{Gr}_d(\mathbb{R}^m); Q_n)$ when m is even

Proof of Theorem 1.9(a) We are going to show that $Q_n(s_\lambda) = 0$ for each Schubert basis element s_λ in degree $d(m-d) - 2^{n+1} + 1$. Since $s_{(d^{(m-d)})}$ is the only class in degree $d(m-d)$,

$$Q_n(s_\lambda) = d_{\lambda(d^{(m-d)})} s_{(d^{(m-d)})},$$

where $d_{\lambda(d^{(m-d)})}$ is given by (3-2). We must only consider λ such that $(d^{(m-d)})/\lambda$ is a broken border strip. As $(d^{(m-d)})$ is a $d \times (m-d)$ grid the complement $(d^{(m-d)})/\lambda$ is always connected and so if $(d^{(m-d)})/\lambda$ is a broken border strip it must be, in particular, a border strip. If $(d^{(m-d)})/\lambda$ is a border strip, then it must be one of three types:

(1) $(d^{(m-d)})/\lambda$ is the last row of $(d^{(m-d)})$,

(2) $(d^{(m-d)})/\lambda$ is the last column of $(d^{(m-d)})$,

(3) $(d^{(m-d)})/\lambda$ is the union of the last row and last column of $(d^{(m-d)})$.

We will show that $d_{\lambda(d^{(m-d)})} = 0$ in each of these cases. As m was assumed to be even, the content of the right most bottom box of $(d^{(m-d)})$ is also even.

1. For the first case, there is just one sharp corner, namely the left most box, and there are no dull corners. Since the strip is of odd length, namely, $2^{n+1} - 1$, the leftmost box and the rightmost box have the same content modulo two. Hence, the content of this sharp corner is zero modulo two, and so $d_{\lambda(d^{(m-d)})} = 0$.
2. For the second case, the argument is exactly the same, but with the sharp corner on the top.
3. For the third case, the content of the sharp corner on the bottom left and the content of the sharp corner on the top right agree modulo two, because the border strip is of odd length. There is one dull corner in the bottom right and it is zero modulo two. Thus, the two sharp corners cancel and the dull corner contributes nothing.

Thus, in all cases $Q_n(s_\lambda) = 0$ for s_λ in degree $d(m-d) - 2^{n+1} + 1$. This completes the proof that the top class is not in the image of Q_n for even m. □

Proof of Theorem 1.9(b) We wish to prove that, when m is even, the chain complexes $\big(\widetilde{H}^*(C_d(\mathbb{R}^m)); Q_n\big)$ and $\big(H^{d(m-d)-*}(\mathrm{Gr}_{d-1}(\mathbb{R}^{m-1})); Q_n\big)$ are dual.

By Proposition 3.4, $\big(\widetilde{H}^{*+m-d}(C_d(\mathbb{R}^m)); Q_n\big)$ is isomorphic to the chain complex

$$\big(H^*(\mathrm{Gr}_{d-1}(\mathbb{R}^{m-1})); \widehat{Q}_n\big),$$

where we recall that $\widehat{Q}_n(y) = Q_n(y) + y\alpha_n$, and that $\alpha_n \bar{w}_{m-d} = Q_n(\bar{w}_{m-d}) \in H^*(\mathrm{Gr}_d(\mathbb{R}^m); \mathbb{Z}/2)$.

So we need to check that the chain complexes

$$\big(H^*(\mathrm{Gr}_{d-1}(\mathbb{R}^{m-1})); \widehat{Q}_n\big) \quad \text{and} \quad \big(H^{(d-1)(m-d)-*}(\mathrm{Gr}_{d-1}(\mathbb{R}^{m-1})); Q_n\big)$$

are dual. This means we need to show that, if $x, y \in H^*(\mathrm{Gr}_{d-1}(\mathbb{R}^{m-1}); \mathbb{Z}/2)$ satisfy

$$|x| + |y| + |Q_n| = (d-1)(m-1),$$

then

$$Q_n(x)y = x\widehat{Q}_n(y).$$

By Theorem 1.9(a), we know that

$$Q_n(xy\bar{w}_{m-d}) = 0 \in H^{d(m-d)}(\mathrm{Gr}_d(\mathbb{R}^m); \mathbb{Z}/2).$$

Thus, in $H^{d(m-d)}(\mathrm{Gr}_d(\mathbb{R}^m);\mathbb{Z}/2)$,

$$\begin{aligned}0 &= Q_n(xy\bar{w}_{m-d})\\ &= Q_n(x)y\bar{w}_{m-d}+xQ_n(y)\bar{w}_{m-d}+xyQ_n(\bar{w}_{m-d})\\ &= Q_n(x)y\bar{w}_{m-d}+xQ_n(y)\bar{w}_{m-d}+xy\alpha_n\bar{w}_{m-d}\\ &= (Q_n(x)y+xQ_n(y)+xy\alpha_n)\bar{w}_{m-d}\\ &= (Q_n(x)y+x\hat{Q}_n(y))\bar{w}_{m-d},\end{aligned}$$

and we conclude that $0=Q_n(x)y+x\hat{Q}_n(y)\in H^{(d-1)(m-d)}(\mathrm{Gr}_{d-1}(\mathbb{R}^{m-1});\mathbb{Z}/2)$. □

Proof of Theorem 1.9(c) Recall that $k_{Q_n}(X)$ denotes the rank of the Q_n–homology $H^*(X;Q_n)$. Similarly, let $\bar{k}_{Q_n}(X)$ denote the rank of $\widetilde{H}^*(X;Q_n)$.

Let $m=2^{n+1}-\epsilon+2l$ with $\epsilon=0$ or 1, and $l\geq 0$. Let

$$k_n^G(d,m)=\sum_{i=0}^{\lfloor d/2\rfloor}\binom{2^{n+1}-\epsilon}{d-2i}\binom{l}{i}.$$

We start with the first part of Theorem 1.9(c). This asserts that, when m is even, if we assume that

$$k_{Q_n}(\mathrm{Gr}_d(\mathbb{R}^{m-1}))=k_n^G(d,m-1)\quad\text{and}\quad k_{Q_n}(\mathrm{Gr}_{d-1}(\mathbb{R}^{m-1}))=k_n^G(d-1,m-1),$$

then we can conclude that $k_{Q_n}(\mathrm{Gr}_d(\mathbb{R}^m))=k_n^G(d,m)$.

Theorem 1.2 tells us that $k_{Q_n}(\mathrm{Gr}_d(\mathbb{R}^m))\geq k_n^G(d,m)$.

Since we have a short exact sequence

$$0\to\widetilde{H}^*(C_d(\mathbb{R}^m))\to H^*(\mathrm{Gr}_d(\mathbb{R}^m))\to H^*(\mathrm{Gr}_d(\mathbb{R}^{m-1}))\to 0,$$

we see that

$$k_{Q_n}(\mathrm{Gr}_d(\mathbb{R}^{m-1}))+\bar{k}_{Q_n}(C_d(\mathbb{R}^m))\geq k_{Q_n}(\mathrm{Gr}_d(\mathbb{R}^m)),$$

with equality if and only if the associated long exact Q_n–homology sequence is still short exact.

Since m is even, Theorem 1.9(b) applies, and tells us that $\bar{k}_{Q_n}(C_d(\mathbb{R}^m))=k_{Q_n}(\mathrm{Gr}_{d-1}(\mathbb{R}^{m-1}))$.

Putting this all together, under our assumptions,

$$k_n^G(d,m-1)+k_n^G(d-1,m-1)\geq k_{Q_n}(\mathrm{Gr}_d(\mathbb{R}^m))\geq k_n^G(d,m).$$

That these would be, in fact, equalities, follows from the next lemma.

Lemma 4.1 *If $m=2^{n+1}+2l$ with $l\geq 0$, then*

$$k_n^G(d,m-1)+k_n^G(d-1,m-1)=k_n^G(d,m).$$

Proof We compute

$$\begin{aligned} k_n^G(d,m-1)+k_n^G(d-1,m-1) &= \sum_i \Big[\binom{2^{n+1}-1}{d-2i}+\binom{2^{n+1}-1}{d-2i-1}\Big]\binom{l}{i} \\ &= \sum_i \binom{2^{n+1}}{d-2i}\binom{l}{i} \\ &= k_n^G(d,m). \end{aligned}$$ □

When all of this happens, we then see that the Q_n–homology long exact sequence really is still short exact, and also that the $K(n)$–AHSS must collapse for these three spaces. Thus there is also a short exact sequence

$$0 \to \widetilde{K}(n)^*(C_d(\mathbb{R}^m)) \xrightarrow{p^*} K(n)^*(\mathrm{Gr}_d(\mathbb{R}^m)) \xrightarrow{i^*} K(n)^*(\mathrm{Gr}_d(\mathbb{R}^{m-1})) \to 0.$$

Finally, the top cohomology class in $H^{d(m-d)}(\mathrm{Gr}_d(\mathbb{R}^m);\mathbb{Z}/2)$ will be a permanent cycle in the AHSS computing $K(n)^*(\mathrm{Gr}_d(\mathbb{R}^m))$ and thus also in the AHSS computing $k(n)^*(\mathrm{Gr}_d(\mathbb{R}^m))$, and this is equivalent to saying that $\mathrm{Gr}_d(\mathbb{R}^m)$ is $k(n)$–oriented. □

5 Results about $H^*(\mathrm{Gr}_d(\mathbb{R}^m);Q_n)$ when $d=2$

In this section we present our results about the Q_n–homology of $\mathrm{Gr}_2(\mathbb{R}^m)$, with the focus on understanding the case when m has the form $2^{n+1}-1+2l$.

To begin with, we know that

- $H^*(\mathrm{Gr}_2(\mathbb{R}^m);\mathbb{Z}/2)=\mathbb{Z}/2[w_1,w_2]/(\bar{w}_{m-1},\bar{w}_m)$;
- in $H^*(\mathrm{Gr}_2(\mathbb{R}^m);\mathbb{Z}/2)$, the ideal $\widetilde{H}^*(C_2(\mathbb{R}^m);\mathbb{Z}/2)$ has an additive basis $\{w_1^i\bar{w}_{m-2} \mid 0\le i\le m-2\}$.

Now we collect results that hold in $H^*(\mathrm{Gr}_2(\mathbb{R}^\infty);\mathbb{Z}/2)$.

Lemma 5.1 *In* $H^*(\mathrm{Gr}_2(\mathbb{R}^\infty);\mathbb{Z}/2)$,

(a) $\bar{w}_0=1$, $\bar{w}_1=w_1$, *and, recursively*, $\bar{w}_k=w_1\bar{w}_{k-1}+w_2\bar{w}_{k-2}$;

(b) $w_2^j\bar{w}_k=\sum_i\binom{j}{i}w_1^{j-i}\bar{w}_{k+j+i}$;

(c) $\bar{w}_k=\sum_j\binom{k-j}{j}w_1^{k-2j}w_2^j$;

(d) $\bar{w}_{2^b-1}=w_1^{2^b-1}$ *for all* $b\ge 0$;

(e) $\bar{w}_{2^b-2}=\sum_{c=0}^{b-1}w_1^{2^b-2^{c+1}}w_2^{2^c-1}$ *for all* $b\ge 1$.

Proof The homogeneous components of the equation $0=(1+w_1+w_2)(1+\bar{w}_1+\bar{w}_2+\cdots)$ give statement (a).

Statement (b) is proved by induction on j. The case when $j=0$ is trivial, and statement (a) rewrites as $w_2\bar{w}_k = w_1\bar{w}_{k+1}+\bar{w}_{k+2}$, which is the case when $j=1$. One then computes

$$\begin{aligned} w_2^j\bar{w}_k &= w_2(w_2^{j-1}\bar{w}_k)\\ &= \sum_i \binom{j-1}{i} w_1^{j-1-i} w_2\bar{w}_{k+j-1+i}\\ &= \sum_i \binom{j-1}{i}[w_1^{j-i}\bar{w}_{k+j+i} + w_1^{j-1-i}\bar{w}_{k+j+i+1}]\\ &= \sum_i \left[\binom{j-1}{i}+\binom{j-1}{i-1}\right] w_1^{j-i}\bar{w}_{k+j+i}\\ &= \sum_i \binom{j}{i} w_1^{j-i}\bar{w}_{k+j+i}. \end{aligned}$$

For (c), note that $\bar{w}_k$ is the homogeneous component of degree k in

$$\bar{w} = (1+w_1+w_2)^{-1} = \sum_{t=0}^{\infty}(w_1+w_2)^t.$$

Statement (d) follows from (c):

$$\bar{w}_{2^b-1} = \sum_j \binom{2^b-1-j}{j} w_1^{k-2j} w_2^j = w_1^{2^b-1},$$

using that $\binom{2^b-1-j}{j} \equiv 1 \bmod 2$ only if $j=0$.

Similarly, statement (e) follows from (c):

$$\bar{w}_{2^b-2} = \sum_j \binom{2^b-2-j}{j} w_1^{k-2j} w_2^j = \sum_{c=0}^{b-1} w_1^{2^b-2^{c+1}} w_2^{2^c-1},$$

using that $\binom{2^b-2-j}{j} \equiv 1 \bmod 2$ if and only if $j=2^c-1$ with $0\le j\le b-1$. □

Now we determine the action of Q_n on various classes.

Lemma 5.2 *In* $H^*(\mathrm{Gr}_2(\mathbb{R}^\infty);\mathbb{Z}/2)$, $Q_n(w_1) = w_1^{2^{n+1}} = w_1\bar{w}_{2^{n+1}-1}$.

Proof The first equality here was already noted in the proof of Proposition 3.5, and the second follows from Lemma 5.1(d). □

Lemma 5.3 *In* $H^*(\mathrm{Gr}_2(\mathbb{R}^\infty);\mathbb{Z}/2)$,

$$Q_n(w_2) = \sum_{c=0}^{n} w_1^{2^{n+1}-2^{c+1}+1} w_2^{2^c} = w_1w_2\bar{w}_{2^{n+1}-2}.$$

Proof The second equality here follows from Lemma 5.1(e), so we just need to check the first. We do this by induction on n, where the $n=0$ case is the easily checked: $Q_0(w_2)=\mathrm{Sq}^1(w_2)=w_1w_2$.

Before proceeding with the inductive step, we make two observations.

The first is that for $n\geq 1$, $Q_n(w_2)=\mathrm{Sq}^{2^n}Q_{n-1}(w_2)$ because the other term, $Q_{n-1}\mathrm{Sq}^{2^n}(w_2)$, will be zero. This is clear if $n\geq 2$ as then $\mathrm{Sq}^{2^n}(w_2)=0$, and when $n=1$, we observe that $Q_0\mathrm{Sq}^2(w_2)=\mathrm{Sq}^1(w_2^2)=0$.

The second observation is that $\mathrm{Sq}(w_2)=w_2(1+w_1+w_2)$, so $\mathrm{Sq}(w_2^{2^c})=w_2^{2^c}(1+w_1^{2^c}+w_2^{2^c})$, and thus

$$\mathrm{Sq}^j(w_2^{2^c})=\begin{cases} w_2^{2^c} & \text{if } j=0,\\ w_1^{2^c}w_2^{2^c} & \text{if } j=2^c,\\ w_2^{2^{c+1}} & \text{if } j=2^{c+1},\\ 0 & \text{otherwise.}\end{cases}$$

Now we check the inductive step of our proof.

$$\begin{aligned} Q_n(w_2) &= \mathrm{Sq}^{2^n}Q_{n-1}(w_2) && \text{(by our first observation)}\\ &= \sum_{c=0}^{n-1}\mathrm{Sq}^{2^n}(w_1^{2^n-2^{c+1}+1}w_2^{2^c}) && \text{(by inductive hypothesis)}\\ &= \sum_{c=0}^{n-1}\sum_j \mathrm{Sq}^{2^n-j}(w_1^{2^n-2^{c+1}+1})\mathrm{Sq}^j(w_2^{2^c}). \end{aligned}$$

By our second observation, the only possible nonzero terms in this double sum are when $j=0,2^c,2^{c+1}$. The terms with $j=0$ are all zero, as $\mathrm{Sq}^{2^n}(w_1^{2^n-2^{c+1}+1})=0$ by the unstable condition. Similarly, the only nonzero term with $j=2^c$ is the term $w_1^{2^{n+1}-1}w_2$, when $c=0$. Finally, one gets $w_1^{2^{n+1}-2^{c+2}+1}w_2^{2^{c+1}}$ when $j=2^{c+1}$ for all $0\leq c\leq n-1$. One is left with

$$Q_n(w_2)=\sum_{c=0}^{n} w_1^{2^{n+1}-2^{c+1}+1}w_2^{2^c},$$

completing our induction. □

Remark 5.4 The referee has pointed out that the first equality in the last lemma appears in [13, page 508].[1]

We now turn our attention to the behavior of Q_n on $\widetilde{H}^*(C_2(\mathbb{R}^m);\mathbb{Z}/2)$.

Lemma 5.5 *In* $\widetilde{H}^*(C_2(\mathbb{R}^m);\mathbb{Z}/2)$, $Q_n(\bar{w}_{m-2})=w_1^{2^{n+1}-1}\bar{w}_{m-2}$.

Proof By Corollary 3.7, $Q_n(\bar{w}_{m-2})=w_{Q_n}(\gamma_1)\bar{w}_{m-2}$, where $\gamma_1\to \mathrm{Gr}_1(\mathbb{R}^{m-1})$ is the canonical line bundle, and Proposition 3.5 tells us that $w_{Q_n}(\gamma_1)=w_1^{2^{n+1}-1}$. □

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[1] There is a slight misprint, and a proof is just hinted at.

Remark 5.6 This lemma also admits a proof using the Schubert cell perspective; see [10, Lemma 4.9.13].

As Q_n is a derivation, the lemma, together with the calculation $Q_n(w_1) = w_1^{2^{n+1}}$, allows one to easily compute the Q_n–homology of $C_2(\mathbb{R}^m)$. What results is the following.

Proposition 5.7 (a) *In $\widetilde{H}^*(C_2(\mathbb{R}^m); \mathbb{Z}/2)$,*

$$Q_n(w_1^i \bar{w}_{m-2}) = \begin{cases} w_1^{2^{n+1}-1+i} \bar{w}_{m-2} & \text{if } i \text{ is even,} \\ 0 & \text{if } i \text{ is odd.} \end{cases}$$

(b) *If $m \le 2^{n+1}$, then Q_n acts as zero on $\widetilde{H}^*(C_2(\mathbb{R}^m); \mathbb{Z}/2)$. Thus $\tilde{k}_{Q_n}(C_2(\mathbb{R}^m)) = m - 1$.*

(c) *If $m > 2^{n+1}$ and is even, then the classes*

$$\{w_1^{2j-1} \bar{w}_{m-2} \mid 1 \le j \le 2^n - 1\} \quad \text{and} \quad \{w_1^{m-2j} \bar{w}_{m-2} \mid 1 \le j \le 2^n\}$$

represent the Q_n–homology classes. Thus $\tilde{k}_{Q_n}(C_2(\mathbb{R}^m)) = 2^{n+1} - 1$.

(d) *If $m > 2^{n+1}$ and is odd, then the classes*

$$\{w_1^{2j-1} \bar{w}_{m-2} \mid 1 \le j \le 2^n - 1\} \quad \text{and} \quad \{w_1^{m-1-2j} \bar{w}_{m-2} \mid 1 \le j \le 2^n - 1\}$$

represent the Q_n–homology classes. Thus $\tilde{k}_{Q_n}(C_2(\mathbb{R}^m)) = 2^{n+1} - 2$.

Proof of Theorem 1.10 Let $m = 2^{n+1} + 1 + 2l$. We need to prove that the map

$$\widetilde{H}^*(C_2(\mathbb{R}^m); Q_n) \xrightarrow{p^*} H^*(\mathrm{Gr}_2(\mathbb{R}^m); Q_n)$$

is zero; ie we need to show that representatives of the Q_n–homology classes in $\widetilde{H}^*(C_2(\mathbb{R}^m); \mathbb{Z}/2)$ are in the image of Q_n when regarded in $H^*(\mathrm{Gr}_2(\mathbb{R}^m); \mathbb{Z}/2)$.

By Proposition 5.7(d), these representatives are in two families,

$$w_1^{1+2j} \bar{w}_{2^{n+1}-1+2l} \quad \text{and} \quad w_1^{2l+2+2j} \bar{w}_{2^{n+1}-1+2l},$$

both with $0 \le j \le 2^n - 2$.

If we can find $a, b \in H^*(\mathrm{Gr}_2(\mathbb{R}^m; \mathbb{Z}/2)$ such that

$$Q_n(a) = w_1 \bar{w}_{2^{n+1}-1+2l} \quad \text{and} \quad Q_n(b) = w_1^{2l+2} \bar{w}_{2^{n+1}-1+2l},$$

we will be done, as then

$$Q_n(w_1^{2j} a) = w_1^{1+2j} \bar{w}_{2^{n+1}-1+2l} \quad \text{and} \quad Q_n(w_1^{2j} b) = w_1^{2l+2+2j} \bar{w}_{2^{n+1}-1+2l}.$$

Thus the next two propositions finish the proof. □

Proposition 5.8 *In $H^*(\mathrm{Gr}_2(\mathbb{R}^\infty); \mathbb{Z}/2)$,*

$$Q_n(w_1 \bar{w}_{2l}) = w_1 \bar{w}_{2^{n+1}-1+2l}.$$

Proposition 5.9 *In $H^*(\mathrm{Gr}_2(\mathbb{R}^{2^{n+1}+1+2l});\mathbb{Z}/2)$,*

$$Q_n(w_2^{2l+1}) = w_1^{2l+2}\bar{w}_{2^{n+1}-1+2l}.$$

Before proving these, we first run through how Theorem 1.10 leads to the proof of Theorem 1.8.

Proof of Theorem 1.8 Our goal is to show that if $m = 2^{n+1}-\epsilon+2l$ with $\epsilon = 0$ or 1, and $l \geq 0$, then $k_{Q_n}(\mathrm{Gr}_2(\mathbb{R}^m)) = \binom{2^{n+1}-\epsilon}{2} + l$, the lower bound coming from Theorem 1.2.

We prove this by induction on m, with the two cases when $l = 0$ already covered by Theorem 1.1. The case when m is even is covered by Theorem 1.9(c), as we know our calculations are right for $(n,1,m-1)$, and by induction we can assume the theorem for $(n,2,m-1)$.

Suppose m is odd, so $\epsilon = 1$ and $m-1 = 2^{n+1}+2(l-1)$. By induction, we can assume that $k_{Q_n}(\mathrm{Gr}_2(\mathbb{R}^{m-1})) = \binom{2^{n+1}}{2} + (l-1)$. Then

$$\begin{aligned} k_{Q_n}(\mathrm{Gr}_2(\mathbb{R}^m)) &= k_{Q_n}(\mathrm{Gr}_2(\mathbb{R}^{m-1})) - \bar{k}_{Q_n}(C_2(\mathbb{R}^m)) && \text{(by Theorem 1.10)} \\ &= \binom{2^{n+1}}{2} + (l-1) - (2^{n+1}-2) && \text{(by Proposition 5.7(d))} \\ &= \binom{2^{n+1}-1}{2} + l. \end{aligned}$$

□

It remains to prove Propositions 5.8 and 5.9.

Proof of Proposition 5.8 We prove by induction on l that

$$Q_n(w_1\bar{w}_{2l}) = w_1\bar{w}_{2^{n+1}-1+2l}$$

holds in $H^*(\mathrm{Gr}_2(\mathbb{R}^\infty);\mathbb{Z}/2)$.

We start the induction by checking both the $l = 0$ and $l = 1$ cases.

When $l = 0$, this reads $Q_n(w_1) = w_1\bar{w}_{2^{n+1}}$, which was proved in Lemma 5.2.

We check the $l = 1$ case using both Lemmas 5.2 and 5.3,

$$\begin{aligned} Q_n(w_1\bar{w}_2) &= Q_n(w_1(w_2+w_1^2)) \\ &= Q_n(w_1w_2+w_1^3) \\ &= Q_n(w_1)w_2 + w_1Q_n(w_2) + w_1^2Q_n(w_1) \\ &= w_1w_2\bar{w}_{2^{n+1}-1} + w_1^2w_2\bar{w}_{2^{n+1}-2} + w_1^3\bar{w}_{2^{n+1}-1} \\ &= w_1[w_2\bar{w}_{2^{n+1}-1} + w_1(w_2\bar{w}_{2^{n+1}-2} + w_1\bar{w}_{2^{n+1}-1})] \\ &= w_1[w_2\bar{w}_{2^{n+1}-1} + w_1\bar{w}_{2^{n+1}}] \\ &= w_1\bar{w}_{2^{n+1}+1}. \end{aligned}$$

For the inductive case, we use the identity $\bar{w}_k = w_2^2\bar{w}_{k-4} + w_1^2\bar{w}_{k-2}$ which holds for all $k \geq 4$. Then

$$\begin{aligned}
Q_n(w_1\bar{w}_{2l}) &= Q_n(w_1w_2^2\bar{w}_{2(l-2)} + w_1^3\bar{w}_{2(l-1)}) \\
&= w_2^2Q_n(w_1\bar{w}_{2(l-2)}) + w_1^2Q_n(w_1\bar{w}_{2(l-1)}) \\
&= w_1w_2^2\bar{w}_{2^{n+1}+2(l-2)-1} + w_1^3\bar{w}_{2^{n+1}+2(l-1)-1} \\
&= w_1[w_2^2\bar{w}_{2^{n+1}+2(l-2)-1} + w_1^2\bar{w}_{2^{n+1}+2(l-1)-1}] \\
&= w_1\bar{w}_{2^{n+1}+2l-1}. \qquad \square
\end{aligned}$$

Proof of Proposition 5.9 We wish to prove that

$$Q_n(w_2^{2l+1}) = w_1^{2l+2}\bar{w}_{2^{n+1}-1+2l}$$

holds in $H^*(\mathrm{Gr}_2(\mathbb{R}^{2^{n+1}+1+2l}); \mathbb{Z}/2)$.

We begin with a calculation in $H^*(\mathrm{Gr}_2(\mathbb{R}^\infty); \mathbb{Z}/2)$,

$$\begin{aligned}
Q_n(w_2^{2l+1}) &= w_2^{2l}Q_n(w_2) \\
&= w_1w_2^{2l+1}\bar{w}_{2^{n+1}-2} && \text{(using Lemma 5.3)} \\
&= \sum_i \binom{2l+1}{i} w_1^{2l+2-i}\bar{w}_{2^{n+1}+2l-1+i} && \text{(using Lemma 5.1(b)).}
\end{aligned}$$

When we project this sum onto

$$H^*(\mathrm{Gr}_2(\mathbb{R}^{2^{n+1}+1+2l}); \mathbb{Z}/2) = \mathbb{Z}/2[w_1, w_2]/(\bar{w}_k \mid k \geq 2^{n+1} + 2l),$$

only the term with $i = 0$ is not zero. In other words

$$Q_n(w_2^{2l+1}) = w_1^{2l+2}\bar{w}_{2^{n+1}-1+2l}$$

holds in $H^*(\mathrm{Gr}_2(\mathbb{R}^{2^{n+1}+1+2l}); \mathbb{Z}/2)$. $\square$

6 Towards the conjectures

As organized in this paper, we are trying to calculate $H^*(\mathrm{Gr}_d(\mathbb{R}^m); Q_n)$ by induction on m (and d) with two steps:

- calculate $\widetilde{H}^*(C_d(\mathbb{R}^m); Q_n)$, recalling that $C_d(\mathbb{R}^m)$ is the Thom space of a bundle over $\mathrm{Gr}_{d-1}(\mathbb{R}^{m-1})$;
- calculate $\delta\colon H^*(\mathrm{Gr}_d(\mathbb{R}^{m-1}); Q_n) \to \widetilde{H}^{*+2^{n+1}-1}(C_d(\mathbb{R}^m); Q_n)$.

When m is even, Theorem 1.9 says we can carry through with this plan. In this section we speculate about how things might go when m is odd.

Firstly, we have the analogues of Theorems 1.1 and 1.2 for

$$\bar{k}_n(C_d(\mathbb{R}^m)) = \dim_{K(n)_*} \widetilde{K}(n)^*(C_d(\mathbb{R}^m)).$$

Theorem 6.1 *If $m \le 2^{n+1}$, then $\bar{k}_n(C_d(\mathbb{R}^m)) = \binom{m-1}{d-1}$.*

Proof Theorem 1.1 implies that, if $m \le 2^{n+1}$, the inclusion $\mathrm{Gr}_d(\mathbb{R}^{m-1}) \to \mathrm{Gr}_d(\mathbb{R}^m)$ induces an inclusion $K(n)_*(\mathrm{Gr}_d(\mathbb{R}^{m-1})) \to K(n)_*(\mathrm{Gr}_d(\mathbb{R}^m))$, as this is true in mod p homology. Thus

$$\bar{k}_n(C_d(\mathbb{R}^m)) = k_n(\mathrm{Gr}_d(\mathbb{R}^m)) - k_n(\mathrm{Gr}_d(\mathbb{R}^{m-1})) = \binom{m}{d} - \binom{m-1}{d} = \binom{m-1}{d-1}. \qquad \square$$

Theorem 6.2 *Let $m = 2^{n+1} - \epsilon + 2l$ with $\epsilon = 0$ or 1, and $l \ge 0$. Then*

$$k_n(C_d(\mathbb{R}^m)) \ge \sum_{i=0}^{\lfloor d/2 \rfloor} \binom{2^{n+1}-1-\epsilon}{d-1-2i}\binom{l}{i}.$$

Proof The proof is similar to the proof of Theorem 1.2, with a little tweak.

If V is a real representation of C_4, and W is a subrepresentation, let $C_d(V, W)$ denote the cofiber of the inclusion $\mathrm{Gr}_d(W) \hookrightarrow \mathrm{Gr}_d(V)$; this is a based C_4 space.

If $\dim V = m$ and $\dim W = m-1$, then $C_d(\mathbb{R}^m) = C_d(V, W)$ and thus $\bar{k}_n(C_d(\mathbb{R}^m)) \ge \bar{k}_n(C_d(V, W)^{C_4})$, by our chromatic fixed point theorem, Theorem 1.4. Furthermore, $C_d(V, W)^{C_4}$ will be the cofiber of the inclusion $\mathrm{Gr}_d(W)^{C_4} \hookrightarrow \mathrm{Gr}_d(V)^{C_4}$.

Now we choose V and W. Recall that L_1 and L_2 were the one-dimensional real representations of C_4 and R was the two-dimensional irreducible. We let $V = L_1^{2^n} \oplus L_2^{2^n-\epsilon} \oplus R^l$ and $W = L_1^{2^n-1} \oplus L_2^{2^n-\epsilon} \oplus R^l$.

Proposition 2.1 tells us that

$$\mathrm{Gr}_d(V)^{C_4} = \coprod_{j+k+2i=d} \mathrm{Gr}_j(\mathbb{R}^{2^n}) \times \mathrm{Gr}_k(\mathbb{R}^{2^n-\epsilon}) \times \mathrm{Gr}_i(\mathbb{C}^l)$$

and

$$\mathrm{Gr}_d(W)^{C_4} = \coprod_{j+k+2i=d} \mathrm{Gr}_j(\mathbb{R}^{2^n-1}) \times \mathrm{Gr}_k(\mathbb{R}^{2^n-\epsilon}) \times \mathrm{Gr}_i(\mathbb{C}^l),$$

so

$$C_d(V, W)^{C_4} = \bigvee_{j+k+2i=d} C_j(\mathbb{R}^{2^n}) \wedge \mathrm{Gr}_k(\mathbb{R}^{2^n-\epsilon})_+ \wedge \mathrm{Gr}_i(\mathbb{C}^l)_+.$$

Thus,

$$\begin{aligned}
\bar{k}_n(C_d(\mathbb{R}^m)) &\ge \sum_{j+k+2i=d} \bar{k}_{n-1}(C_j(\mathbb{R}^{2^n}))k_{n-1}(\mathrm{Gr}_k(\mathbb{R}^{2^n-\epsilon}))k_{n-1}(\mathrm{Gr}_i(\mathbb{C}^l)) \\
&= \sum_{j+k+2i=d} \binom{2^n-1}{j-1}\binom{2^n-\epsilon}{k}\binom{l}{i} \quad \text{(using Theorems 6.1 and 1.1)} \\
&= \sum_i \Bigg[\sum_{j+k=d-2i} \binom{2^n-1}{j-1}\binom{2^n-\epsilon}{k}\Bigg]\binom{l}{i} \\
&= \sum_i \binom{2^{n+1}-1-\epsilon}{d-1-2i}\binom{l}{i}.
\end{aligned} \qquad \square$$

Conjecture 6.3 *Equality holds in Theorem 6.2.*

As before, this would be implied by a conjectural calculation of the Q_n homology of $C_d(\mathbb{R}_n)$.

Conjecture 6.4 *Let $m = 2^{n+1} - \epsilon + 2l$ with $\epsilon = 0$ or 1, and $l \geq 0$. Then*

$$\bar{k}_{Q_n}(C_d(\mathbb{R}^m)) = \sum_i \binom{2^{n+1}-1-\epsilon}{d-1-2i}\binom{l}{i}.$$

Our various conjectures imply a conjecture about the behavior of the boundary map

$$\delta\colon H^*(\mathrm{Gr}_d(\mathbb{R}^{m-1}); Q_n) \to \widetilde{H}^{*+2^{n+1}-1}(C_d(\mathbb{R}^m); Q_n)$$

when $m = 2^{n+1} - \epsilon + 2l$. Let $k_n^{\delta}(d,m)$ denote the dimension of the image of this map.

Conjecture 1.7 says that $k_{Q_n}(\mathrm{Gr}_d(\mathbb{R}^m)) = k_n^G(d,m)$, where

$$k_n^G(d,m) = \sum_i \binom{2^{n+1}-\epsilon}{d-2i}\binom{l}{i}.$$

Conjecture 6.4 similarly says that $\bar{k}_{Q_n}(C_d(\mathbb{R}^m)) = \bar{k}_n^C(d,m)$, where

$$\bar{k}_n^C(d,m) = \sum_i \binom{2^{n+1}-1-\epsilon}{d-1-2i}\binom{l}{i}.$$

If these conjectures are true, then the exactness of the Q_n–homology long exact sequence would imply that

$$k_n^G(d,m) + 2k_n^{\delta}(d,m) = k_n^G(d,m-1) + \bar{k}_n^C(d,m),$$

so that

$$k_n^{\delta}(d,m) = \tfrac{1}{2}\big[k_n^G(d,m-1) + \bar{k}_n^C(d,m) - k_n^G(d,m)\big].$$

As expected, the right hand side here is zero if m is even, ie $\epsilon = 0$.

When m is odd, so $\epsilon = 1$, the right hand side is not zero, but can be rearranged as in the following lemma.

Lemma 6.5 *If $m = 2^{n+1} - 1 + 2l$ and $l > 0$, then*

$$\frac{1}{2}\big[k_n^G(d,m-1) + \bar{k}_n^C(d,m) - k_n^G(d,m)\big] = \sum_i \binom{2^{n+1}-2}{d-1-2i}\binom{l-1}{i}.$$

Proof We expand $k_n^G(d,m-1)$:

$$\begin{aligned} k_n^G(d,m-1) &= \sum_i \binom{2^{n+1}}{d-2i}\binom{l-1}{i} \\ &= \sum_i \bigg[\binom{2^{n+1}-2}{d-2i} + 2\binom{2^{n+1}-2}{d-1-2i} + \binom{2^{n+1}-2}{d-2-2i}\bigg]\binom{l-1}{i}. \end{aligned}$$

We rewrite $k_n^G(d,m)-\bar{k}_n^C(d,m)$:

$$\begin{aligned}
k_n^G(d,m)-\bar{k}_n^C(d,m) &= \sum_i \biggl[\binom{2^{n+1}-1}{d-2i}-\binom{2^{n+1}-2}{d-1-2i}\biggr]\binom{l}{i} \\
&= \sum_i \binom{2^{n+1}-2}{d-2i}\binom{l}{i} \\
&= \sum_i \binom{2^{n+1}-2}{d-2i}\biggl[\binom{l-1}{i}+\binom{l-1}{i-1}\biggr] \\
&= \sum_i \biggl[\binom{2^{n+1}-2}{d-2i}+\binom{2^{n+1}-2}{d-2-2i}\biggr]\binom{l-1}{i}.
\end{aligned}$$

Subtracting our second expression from the first, and dividing by two, proves the lemma. □

Thus we can add the following to our conjectures.

Conjecture 6.6 *If $m=2^{n+1}-1+2l$ and $l>0$, then*

$$k_n^\delta(d,m)=\sum_i \binom{2^{n+1}-2}{d-1-2i}\binom{l-1}{i}.$$

Example 6.7 Suppose that $n=0$, so $m=2l+1$. Conjecture 6.4 predicts that

$$\dim_{\mathbb{Q}} H^*(C_d(\mathbb{R}^{2l+1});\mathbb{Q})=\begin{cases}0 & \text{if } d \text{ is even,}\\ \binom{l}{c} & \text{if } d=2c+1.\end{cases}$$

Similarly, Conjecture 6.6 predicts that

$$k_0^\delta(d,2l+1)=\begin{cases}0 & \text{if } d \text{ is even,}\\ \binom{l-1}{c} & \text{if } d=2c+1.\end{cases}$$

Noting that $k_0^\delta(d,2l+1)$ can be viewed as the dimension of the cokernel of the map

$$i^*\colon H^*(\mathrm{Gr}_d(\mathbb{R}^{2l+1});\mathbb{Q})\to H^*(\mathrm{Gr}_d(\mathbb{R}^{2l});\mathbb{Q}),$$

one can check that our conjectures do correspond to the known behavior of i^* — it takes Pontryagin classes to Pontryagin classes — together with the computations

$$\dim_{\mathbb{Q}} H^*(\mathrm{Gr}_d(\mathbb{R}^m);\mathbb{Q})=\begin{cases}\binom{l}{c} & \text{if } m=2l+1 \text{ and } d=2c \text{ or } 2c+1,\\ 2\binom{l-1}{c} & \text{if } m=2l \text{ and } d=2c+1.\end{cases}$$

Appendix Tables

We present some tables of calculations made by the second author that support Conjecture 1.7. Calculational algorithms used are documented in [10, Appendix B]. For larger Grassmannians the authors used the University of Virginia Rivanna high-performance computing system. The white cells are the conjectured values which have not been checked due to computational limitations. The tables are necessarily symmetric in c and d.

d \ c	1	2	3	4	5	6	7	8	9	10	11
1	2	3	4	3	4	3	4	3	4	3	4
2	3	6	4	7	5	8	6	9	7	10	8
3	4	4	8	7	12	10	16	13	20	16	24
4	3	7	7	14	12	22	18	31	25	41	33
5	4	5	12	12	24	22	40	35	60	51	84
6	3	8	10	22	22	44	40	75	65	116	98
7	4	6	16	18	40	40	80	75	140	126	224
8	3	9	13	31	35	75	75	150	140	266	238
9	4	7	20	25	60	65	140	140	280	266	504
10	3	10	16	41	51	116	126	266	266	532	504
11	4	8	24	33	84	98	224	238	504	504	1008
12	3	11	19	52	70	168	196	434	462	966	966
13	4	9	28	42	112	140	336	378	840	882	1848
14	3	12	22	64	92	232	288	666	750	1632	1716
15	4	10	32	52	144	192	480	570	1320	1452	3168
16	3	13	25	77	117	309	405	975	1155	2607	2871
17	4	11	36	63	180	255	660	825	1980	2277	5148
18	3	14	28	91	145	400	550	1375	1705	3982	4576
19	4	12	40	75	220	330	880	1155	2860	3432	8008
20	3	15	31	106	176	506	726	1881	2431	5863	7007
21	4	13	44	88	264	418	1144	1573	4004	5005	12012
22	3	16	34	122	210	628	936	2509	3367	8372	10374
23	4	14	48	102	312	520	1456	2093	5460	7098	17472
24	3	17	37	139	247	767	1183	3276	4550	11648	14924
25	4	15	52	117	364	637	1820	2730	7280	9828	24752
26	3	18	40	157	287	924	1470	4200	6020	15848	20944
27	4	16	56	133	420	770	2240	3500	9520	13328	34272
28	3	19	43	176	330	1100	1800	5300	7820	21148	28764
29	4	17	60	150	480	920	2720	4420	12240	17748	46512
30	3	20	46	196	376	1296	2176	6596	9996	27744	38760
31	4	18	64	168	544	1088	3264	5508	15504	23256	62016
32	3	21	49	217	425	1513	2601	8109	12597	35853	51357
33	4	19	68	187	612	1275	3876	6783	19380	30039	81396
34	3	22	52	239	477	1752	3078	9861	15675	45714	67032
35	4	20	72	207	684	1482	4560	8265	23940	38304	105336
36	3	23	55	262	532	2014	3610	11875	19285	57589	86317
37	4	21	76	228	760	1710	5320	9975	29260	48279	134596
38	3	24	58	286	590	2300	4200	14175	23485	71764	109802
39	4	22	80	250	840	1960	6160	11935	35420	60214	170016
40	3	25	61	311	651	2611	4851	16786	28336	88550	138138
41	4	23	84	273	924	2233	7084	14168	42504	74382	212520
42	3	26	64	337	715	2948	5566	19734	33902	108284	172040
43	4	24	88	297	1012	2530	8096	16698	50600	91080	263120
44	3	27	67	364	782	3312	6348	23046	40250	131330	212290
45	4	25	92	322	1104	2852	9200	19550	59800	110630	322920
46	3	28	70	392	852	3704	7200	26750	47450	158080	259740
47	4	26	96	348	1200	3200	10400	22750	70200	133380	393120
48	3	29	73	421	925	4125	8125	30875	55575	188955	315315
49	4	27	100	375	1300	3575	11700	26325	81900	159705	475020
50	3	30	76	451	1001	4576	9126	35451	64701	224406	380016

🟦 $cd \leq 2^{n+1} - 1$ ▩ projective spaces 🟨 Theorem 1.8
▨ conjecture verified ☐ conjecture

Table 1: $k_1(\mathrm{Gr}_d(\mathbb{R}^{d+c}))$.

d \ c	1	2	3	4	5	6	7	8	9	10	11
1	2	3	4	5	6	7	8	7	8	7	8
2	3	6	10	15	21	28	22	29	23	30	24
3	4	10	20	35	56	42	64	49	72	56	80
4	5	15	35	70	56	98	78	127	101	157	125
5	6	21	56	56	112	98	176	147	248	203	328
6	7	28	42	98	98	196	176	323	277	480	402
7	8	22	64	78	176	176	352	323	600	526	928
8	7	29	49	127	147	323	323	646	600	1126	1002
9	8	23	72	101	248	277	600	600	1200	1126	2128
10	7	30	56	157	203	480	526	1126	1126	2252	2128
11	8	24	80	125	328	402	928	1002	2128	2128	4256
12	7	31	63	188	266	668	792	1794	1918	4046	4046
13	8	25	88	150	416	552	1344	1554	3472	3682	7728
14	7	32	70	220	336	888	1128	2682	3046	6728	7092
15	8	26	96	176	512	728	1856	2282	5328	5964	13056
16	7	33	77	253	413	1141	1541	3823	4587	10551	11679
17	8	27	104	203	616	931	2472	3213	7800	9177	20856
18	7	34	84	287	497	1428	2038	5251	6625	15802	18304
19	8	28	112	231	728	1162	3200	4375	11000	13552	31856
20	7	35	91	322	588	1750	2626	7001	9251	22803	27555
21	8	29	120	260	848	1422	4048	5797	15048	19349	46904
22	7	36	98	358	686	2108	3312	9109	12563	31912	40118
23	8	30	128	290	976	1712	5024	7509	20072	26858	66976
24	7	37	105	395	791	2503	4103	11612	16666	43524	56784
25	8	31	136	321	1112	2033	6136	9542	26208	36400	93184
26	7	38	112	433	903	2936	5006	14548	21672	58072	78456
27	8	32	144	353	1256	2386	7392	11928	33600	48328	126784
28	7	39	119	472	1022	3408	6028	17956	27700	76028	106156
29	8	33	152	386	1408	2772	8800	14700	42400	63028	169184
30	7	40	126	512	1148	3920	7176	21876	34876	97904	141032
31	8	34	160	420	1568	3192	10368	17892	52768	80920	221952
32	7	41	133	553	1281	4473	8457	26349	43333	124253	184365
33	8	35	168	455	1736	3647	12104	21539	64872	102459	286824
34	7	42	140	595	1421	5068	9878	31417	53211	155670	237576
35	8	36	176	491	1912	4138	14016	25677	78888	128136	365712
36	7	43	147	638	1568	5706	11446	37123	64657	192793	302233
37	8	37	184	528	2096	4666	16112	30343	95000	158479	460712
38	7	44	154	682	1722	6388	13168	43511	77825	236304	380058
39	8	38	192	566	2288	5232	18400	35575	113400	194054	574112
40	7	45	161	727	1883	7115	15051	50626	92876	286930	472934
41	8	39	200	605	2488	5837	20888	41412	134288	235466	708400
42	7	46	168	773	2051	7888	17102	58514	109978	345444	582912
43	8	40	208	645	2696	6482	23584	47894	157872	283360	866272
44	7	47	175	820	2226	8708	19328	67222	129306	412666	712218
45	8	41	216	686	2912	7168	26496	55062	184368	338422	1050640
46	7	48	182	868	2408	9576	21736	76798	151042	489464	863260
47	8	42	224	728	3136	7896	29632	62958	214000	401380	1264640
48	7	49	189	917	2597	10493	24333	87291	175375	576755	1038635
49	8	43	232	771	3368	8667	33000	71625	247000	473005	1511640
50	7	50	196	967	2793	11460	27126	98751	202501	675506	1241136

🟦 $cd \leq 2^{n+1}-1$ ▩ projective spaces 🟨 Theorem 1.8
░ conjecture verified □ conjecture 🟩 Theorem 1.1

Table 2: $k_2(\mathrm{Gr}_d(\mathbb{R}^{d+c}))$.

d \ c	1	2	3	4	5	6	7	8	9	10	11
1	2	3	4	5	6	7	8	9	10	11	12
2	3	6	10	15	21	28	36	45	55	66	78
3	4	10	20	35	56	84	120	165	220	286	364
4	5	15	35	70	126	210	330	495	715	1001	1365
5	6	21	56	126	252	462	792	1287	2002	3003	4368
6	7	28	84	210	462	924	1716	3003	5005	8008	6370
7	8	36	120	330	792	1716	3432	6435	11440	9438	15808
8	9	45	165	495	1287	3003	6435	12870	11440	20878	17810
9	10	55	220	715	2002	5005	11440	11440	22880	20878	38688
10	11	66	286	1001	3003	8008	9438	20878	20878	41756	38688
11	12	78	364	1365	4368	6370	15808	17810	38688	38688	77376
12	13	91	455	1820	3458	9828	12896	30706	33774	72462	72462
13	14	105	560	1470	4928	7840	20736	25650	59424	64338	136800
14	15	120	470	1940	3928	11768	16824	42474	50598	114936	123060
15	16	106	576	1576	5504	9416	26240	35066	85664	99404	222464
16	15	121	485	2061	4413	13829	21237	56303	71835	171239	194895
17	16	107	592	1683	6096	11099	32336	46165	118000	145569	340464
18	15	122	500	2183	4913	16012	26150	72315	97985	243554	292880
19	16	108	608	1791	6704	12890	39040	59055	157040	204624	497504
20	15	123	515	2306	5428	18318	31578	90633	129563	334187	422443
21	16	109	624	1900	7328	14790	46368	73845	203408	278469	700912
22	15	124	530	2430	5958	20748	37536	111381	167099	445568	589542
23	16	110	640	2010	7968	16800	54336	90645	257744	369114	958656
24	15	125	545	2555	6503	23303	44039	134684	211138	580252	800680
25	16	111	656	2121	8624	18921	62960	109566	320704	478680	1279360
26	15	126	560	2681	7063	25984	51102	160668	262240	740920	1062920
27	16	112	672	2233	9296	21154	72256	130720	392960	609400	1672320
28	15	127	575	2808	7638	28792	58740	189460	320980	930380	1383900
29	16	113	688	2346	9984	23500	82240	154220	475200	763620	2147520
30	15	128	590	2936	8228	31728	66968	221188	387948	1151568	1771848
31	16	114	704	2460	10688	25960	92928	180180	568128	943800	2715648
32	15	129	605	3065	8833	34793	75801	255981	463749	1407549	2235597
33	16	115	720	2575	11408	28535	104336	208715	672464	1152515	3388112
34	15	130	620	3195	9453	37988	85254	293969	549003	1701518	2784600
35	16	116	736	2691	12144	31226	116480	239941	788944	1392456	4177056
36	15	131	635	3326	10088	41314	95342	335283	644345	2036801	3428945
37	16	117	752	2808	12896	34034	129376	273975	918320	1666431	5095376
38	15	132	650	3458	10738	44772	106080	380055	750425	2416856	4179370
39	16	118	768	2926	13664	36960	143040	310935	1061360	1977366	6156736
40	15	133	665	3591	11403	48363	117483	428418	867908	2845274	5047278
41	16	119	784	3045	14448	40005	157488	350940	1218848	2328306	7375584
42	15	134	680	3725	12083	52088	129566	480506	997474	3325780	6044752
43	16	120	800	3165	15248	43170	172736	394110	1391584	2722416	8767168
44	15	135	695	3860	12778	55948	142344	536454	1139818	3862234	7184570
45	16	121	816	3286	16064	46456	188800	440566	1580384	3162982	10347552
46	15	136	710	3996	13488	59944	155832	596398	1295650	4458632	8480220
47	16	122	832	3408	16896	49864	205696	490430	1786080	3653412	12133632
48	15	137	725	4133	14213	64077	170045	660475	1465695	5119107	9945915
49	16	123	848	3531	17744	53395	223440	543825	2009520	4197237	14143152
50	15	138	740	4271	14953	68348	184998	728823	1650693	5847930	11596608

🟦 $cd \leq 2^{n+1} - 1$ ▩ projective spaces 🟨 Theorem 1.8

▨ conjecture verified ☐ conjecture 🟩 Theorem 1.1

Table 3: $k_3(\mathrm{Gr}_d(\mathbb{R}^{d+c}))$.

c d	1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10	11
2	3	6	10	15	21	28	36	45	55	66
3	4	10	20	35	56	84	120	165	220	286
4	5	15	35	70	126	210	330	495	715	1001
5	6	21	56	126	252	462	792	1287	2002	3003
6	7	28	84	210	462	924	1716	3003	5005	8008
7	8	36	120	330	792	1716	3432	6435	11440	19448
8	9	45	165	495	1287	3003	6435	12870	24310	43758
9	10	55	220	715	2002	5005	11440	24310	48620	92378
10	11	66	286	1001	3003	8008	19448	43758	92378	184756
11	12	78	364	1365	4368	12376	31824	75582	167960	352716
12	13	91	455	1820	6188	18564	50388	125970	293930	646646
13	14	105	560	2380	8568	27132	77520	203490	497420	1144066
14	15	120	680	3060	11628	38760	116280	319770	817190	1961256
15	16	136	816	3876	15504	54264	170544	490314	1307504	3268760
16	17	153	969	4845	20349	74613	245157	735471	2042975	5311735
17	18	171	1140	5985	26334	100947	346104	1081575	3124550	8436285
18	19	190	1330	7315	33649	134596	480700	1562275	4686825	13123110
19	20	210	1540	8855	42504	177100	657800	2220075	6906900	20030010
20	21	231	1771	10626	53130	230230	888030	3108105	10015005	30045015
21	22	253	2024	12650	65780	296010	1184040	4292145	14307150	44352165
22	23	276	2300	14950	80730	376740	1560780	5852925	20160075	64512240
23	24	300	2600	17550	98280	475020	2035800	7888725	28048800	52240890
24	25	325	2925	20475	118755	593775	2629575	10518300	22789650	75030540
25	26	351	3276	23751	142506	736281	3365856	8625006	31414656	60865896
26	27	378	3654	27405	169911	906192	2799486	11424492	25589136	86455032
27	28	406	4060	31465	201376	767746	3567232	9392752	34981888	70258648
28	29	435	4495	35960	174406	942152	2973892	12366644	28563028	98821676
29	30	465	4960	31930	206336	799676	3773568	10192428	38755456	80451076
30	31	496	4526	36456	178932	978608	3152824	13345252	31715852	112166928
31	32	466	4992	32396	211328	832072	3984896	11024500	42740352	91475576
32	31	497	4557	36953	183489	1015561	3336313	14360813	35052165	126527741
33	32	467	5024	32863	216352	864935	4201248	11889435	46941600	103365011
34	31	498	4588	37451	188077	1053012	3524390	15413825	38576555	141941566
35	32	468	5056	33331	221408	898266	4422656	12787701	51364256	116152712
36	31	499	4619	37950	192696	1090962	3717086	16504787	42293641	158446353
37	32	469	5088	33800	226496	932066	4649152	13719767	56013408	129872479
38	31	500	4650	38450	197346	1129412	3914432	17634199	46208073	176080552
39	32	470	5120	34270	231616	966336	4880768	14686103	60894176	144558582
40	31	501	4681	38951	202027	1168363	4116459	18802562	50324532	194883114
41	32	471	5152	34741	236768	1001077	5117536	15687180	66011712	160245762
42	31	502	4712	39453	206739	1207816	4323198	20010378	54647730	214893492
43	32	472	5184	35213	241952	1036290	5359488	16723470	71371200	176969232
44	31	503	4743	39956	211482	1247772	4534680	21258150	59182410	236151642
45	32	473	5216	35686	247168	1071976	5606656	17795446	76977856	194764678
46	31	504	4774	40460	216256	1288232	4750936	22546382	63933346	258698024
47	32	474	5248	36160	252416	1108136	5859072	18903582	82836928	213668260
48	31	505	4805	40965	221061	1329197	4971997	23875579	68905343	282573603
49	32	475	5280	36635	257696	1144771	6116768	20048353	88953696	233716613
50	31	506	4836	41471	225897	1370668	5197894	25246247	74103237	307819850

🟦 $cd \leq 2^{n+1} - 1$
▨ projective spaces
🟨 Theorem 1.8
░ conjecture verified
☐ conjecture
🟩 Theorem 1.1

Table 4: $k_4(\mathrm{Gr}_d(\mathbb{R}^{d+c}))$.

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